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Degenerations for the Representations of an Equioriented Quiver of Type D_m

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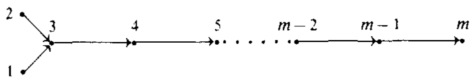
1. INTRODUCTION

We consider the Dynkin diagram D_m, whose vertices are labelled $1, 2, 3, \ldots, m$ and we denote by $\mathbb{Q}_m \equiv (D_m, \text{eq.})$ the graph D_m with the following orientation for its edges:



We call $\mathbb{Q}_m$ an equioriented quiver of type D_m.

Let K be a field. For every

$$d = \begin{pmatrix} d_2 \\ d_1 \end{pmatrix}; d_3, \ldots, d_m \Big) \in N^m$$

we consider the variety $L_d = L_d(D_m, \text{eq.}) := \text{Hom}_K(V_1, V_3) \times \prod_{i=2}^{m-1} \text{Hom}_K(V_i, V_{i+1})$ ($V_j, j = 1, 2, \ldots, m$, a vector space over K of dimension d_j) of all the representations of the oriented graph $\mathbb{Q}_m$ of dimension d, the group $G = G_d := \prod_{i=1}^{m} GL(V_i)$ acts naturally on L_d and the number of orbits of this action is finite, each orbit O_A ($A \in L_d$) corresponding to an isomorphism class $[A]$ of representations (cf. [4–7]).

In this paper we study the problem of the degenerations for the representations of $Q_m = (D_m, \text{eq.})$ of given dimension. Given any orbit $O_A \subset L_d$ we want to characterize the obits $O_B \subset L_d$ such that $O_B \subset \bar{O}_A$ ($\bar{O}_A$ the Zariski closure of O_A), i.e., the degenerations of O_A.

The way we approach the problem is the same as in [2b], where we study the degenerations for the representation of the graph A_m with an arbitrary orientation.

Let

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}; A_3; A_4; \ldots; A_{m-1} \Big) \in L_d$$

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be a representation of dimension d. We use a set N^A of "rank parameters" which have been introduced in [1b] (cf. also Section 2) to prove that $O_B \subset \bar{O}_A$ if and only if $N^B \leqslant N^A$ (cf. Theorem 5.2, the part that states the equality of the orderings $\leqslant_g$ and $\leqslant_r$). The argument we give is combinatorial.

We also prove that if $O_B \subset \bar{O}_A$ and O_B is open in $\bar{O}_A - O_A$, i.e., if O_B is a minimal degeneration of O_A, then there exists a submodule $A' \subset A$ (we think of the representations as modules) such that $B \simeq A' \oplus A/A'$ (cf. Theorem 5.2, the part that states the equality of the orderings $\leqslant_c$ and $\leqslant_g$). The claim is the same we proved for the representations of the Dynkin diagram A_m arbitrarity oriented (cf. [2]).

This phenomenon is not true in general; C. Riedtman has given examples of the following type. Take the oriented graph

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] & \cdot \arrow[loop right, "\beta"] \end{tikzcd}$$

Set $\beta^2 = 0$ and consider the variety of representations of dimension type $(1, 2)$; the generic orbit is indecomposable as well as other special orbits. For the relation $\beta^3 = 0$ and dimension type $(2, 5)$ one has that the generic element is decomposable and there exists a special element that is indecomposable.

The authors believe that the stated property is true for the representations of the Dynkin diagrams A, D, E, independently from the orientation.

Following this idea, our next attempt will be to solve the following problems: (1) characterize the degenerations for the representations of D_n with an arbitrary orientation Ω; and (2) discuss the same type of problems for E_6, E_7, E_8.

Passing from the case treated here to an arbitrary orientation of D_m seems to present te same type of difficulties encountered for the A_m case (cf. [2a, b]); we may remark that the reflection functors do not seem to be able to take care of the orbit structure, except is special cases (cf. also [1a, b]).

Notice that the degeneration $A \mapsto B = A' \oplus A/A'$ is obtained in the following geometric way: we apply a 1-parameter subgroup $\lambda(t)$ to A and we obtain $B = \lim_{t \to 0} \lambda(t) \cdot A$ (cf. [3]).

Conversely if we have a variety of representations of a finitely generated algebra, we can consider its points as particular p-tuples of $q \times q$ matrices $(X_1, ..., X_p)$ and two representations are isomorphic if and only if the p-tuples are simultaneously conjugate under the group $GL(q)$. Now if we have a p-tuple $(X_1, ..., X_p)$ and a 1-parameter subgroup $\lambda(t)$ such that $\lim_{t \to 0} \lambda(t) X_i \lambda(t)^{-1} = Y_i$ is defined for all i, then the representation $(Y_1, ..., Y_p)$ is in the closure of the orbit of $(X_1, ..., X_p)$ and it is obtained as follows. Consider the q-dimensional vector space V over which the X_i's operate.

Decompose V according to the weights of $\lambda(t)$, $V=\bigoplus_{j=1}^{N} V_j$, $\lambda(t)$ acts as t^{n_j} on V_j and $n_1 > n_2 > \cdots > n_N$. For a typical matrix $X = X_i$ write X in block form, $X=(X_{hk})$, $X_{hk}\colon V_k \to V_h$ the corresponding block. Then $\lambda(t)\, X \lambda(t)^{-1} = (t^{n_h - n_k} X_{hk})$ and $\lim_{t\to 0} \lambda(t)\, X\lambda(t)^{-1}$ exists if and only if $X_{hk}=0$ when $h>k$. Thus the subspaces $W_1 = V_1$, $W_2 = V_1 \oplus V_2, \ldots, W_j = V_1 \oplus V_2 \oplus \cdots \oplus V_j$ are stable under X and in the graded space $\bigoplus W_i/W_{i-1}$ the corresponding representation has matrix $Y=\lim_{t\to 0}\lambda(t)\, X\lambda(t)^{-1}$, which is the matrix with only the diagonal blocks appearing.

A problem which should be analyzed is the following: Given an action of a reductive algebraic group G over a verity V, and given two orbits O_1, O_2 in V such that $O_2 \subset \bar{O}_1$, when can we find a one-parameter subgroup $\lambda(t)$ in G such that for a given point $P \in O_1$ we have $\lim_{t\to 0}\lambda(t)P \in O_2$? In the special case in which V is affine and O_2 is a closed orbit this is the Hilbert–Mumford criterion in the form proved by Kempf (cf. [9]).

If O_2 is not closed then this is not always possible and it should be interesting to characterize in a geometric way those pairs of orbits for which such construction is possible.

For our purposes it would be enough to perform the analysis in the case in which O_2 is open in $\bar{O}_1 - O_2$ (i.e., a minimal degeneration).

Thus, in particular, we are trying to establish this last statement for the varieties of representations of the Dynkin diagrams. The results obtained so far are positive for A_m, and for D_m with a special orientation.

2. Indecomposable Representations and Homomorphisms

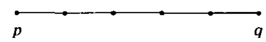
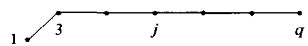
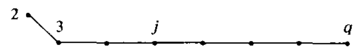
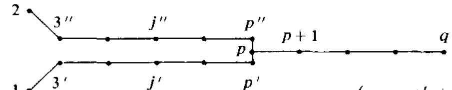
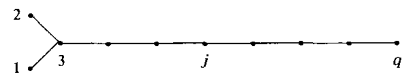
Let us consider the Dynkin diagram D_m equioriented, i.e.:

$$\begin{tikzcd}[row sep=small]
2 \arrow[dr] & & & & & & \\
& 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r, dotted, dash] & m-2 \arrow[r] & m-1 \arrow[r] & m \\
1 \arrow[ur] & & & & & &
\end{tikzcd}$$

where $\Gamma_0 = \{1, 2, 3, \ldots, m\}$ is the set of its vertices. In the set $\{0, 1, 2, 3, \ldots, m\} = \Gamma_0 \cup \{0\}$ we consider the following partial order $\leqslant$ which is the usual order of the set N of natural numbers, except for the integers $0, 1, 2$ which are incomparable. This is the order that we will systematically use in this paper, therefore, for example, the interval $[0, q]$ does not contain the integers $1, 2$; $p+1$ is the first integer following p, therefore $0+1=3$, $1+1=3$. Note that the induced order on the set Γ_0 coincides with the natural order defined by the orientation of the graph, and we have added the integer 0, as it will be useful for the notations of this and the next section.

TABLE 1

List of the Indecomposables of D_n

Notation	Dimension	Graphic description
E_{pq} $3 \leqslant p \leqslant q \leqslant m$	$\begin{matrix} 0 & \\ & 0 \cdots 0\ \underset{p}{\underset{\Vert}{1}} \cdots \underset{q}{\underset{\Vert}{1}}\ 0 \cdots 0 \\ 0 & \end{matrix}$	
E_{1q} $1 \leqslant q \leqslant m$	$\begin{matrix} 0 & \\ & 1 \cdots \underset{q}{\underset{\Vert}{1}}\ 0 \cdots 0 \\ 1 & \end{matrix}$	
E_{2q} $2 \leqslant q \leqslant m$	$\begin{matrix} 1 & \\ & 1 \cdots \underset{q}{\underset{\Vert}{1}}\ 0 \cdots 0 \\ 0 & \end{matrix}$	
F_{pq} $p < q \leqslant m,\ p \in [3, m-1]$	$\begin{matrix} 1 & \\ & 2 \cdots \underset{p}{\underset{\Vert}{2}}\ \underset{p+1}{\underset{\Vert}{1}} \cdots \underset{q}{\underset{\Vert}{1}}\ 0 \cdots 0 \\ 1 & \end{matrix}$	 $(p = p' + p'')$
F_{0q} $3 \leqslant q \leqslant m$	$\begin{matrix} 1 & \\ & 1 \cdots \underset{q}{\underset{\Vert}{1}}\ 0 \cdots 0 \\ 1 & \end{matrix}$	

Recall now that the set of indecomposable representations of D_m is in 1–1 correspondence with the set Δ^+ of the positive roots of D_m, independently from the orientation (cf. [6, 7]). We denote by $E_{pq}, p \leqslant q, p, q = 1, 2, 3,..., m$, the indecomposable representations of D_m which are also indecomposable representations for some subgraph of D_m of type $\mathscr{A}_n$; we denote by F_{pq}, $p < q \leqslant m$, $p \in [0, m-1]$, the remaining ones. In Table 1 we give explicitly the dimensions of the indecompsables and a graphic description, which will be useful in the future. We have to read the graphic description of an indecomposable E_{pq}, resp. F_{pq}, as a representation for the given orientation. The dots j, j', j'' always represent a given basis for the corresponding vector space. In particular the representation F_{pq}, $0 \neq p < q$, is described by: $1 \mapsto 3'$, $2 \mapsto 3''$; $j' \mapsto (j+1)'$, $j'' \mapsto (j+1)''$ if $3 \leqslant j < p$, $p' \mapsto \frac{1}{2}(p+1)$, $p'' \mapsto \frac{1}{2}(p+1)$; $j \mapsto j+1$ if $p < j < q$; $q \mapsto 0$. The interpretation for F_{0q} and E_{pq} is similar and forced by the orientation.

Next we study $\mathrm{Hom}_K(\mathscr{E}_{zw}, \mathscr{E}_{hk})$, where $\mathscr{E}_{zw}$ denotes either E_{zw} or F_{zw} (similarly for $\mathscr{E}_{hk}$). In Table 2 we list the only possible cases for which $\mathrm{Hom}_K(\mathscr{E}_{zw}, \mathscr{E}_{hk}) \neq 0$ and for convenience, we give a graphic description of the corresponding pairs of indecomposables, as in this way it is easier to visualize the mutual position of the indices z, w, h, k. We also given canonical independent generators of $\mathrm{Hom}_K(\mathscr{E}_{zw}, \mathscr{E}_{hk})$ as vector space over K, and we mean that the base vectors of $\mathscr{E}_{zw}$ which are not explicitly mentioned go to zero. We will use these generators in the proof of Proposition 4.1.

Note that we have not considered explicitly the case where z or h is equal to 2, as we have $\mathrm{Hom}_K(E_{1w}, E_{2w}) = 0$ and we can deduce the other possibilities using the permutation σ of the vertices 1, 2 (in fact σ is a graph automorphism which preserves the given orientation).

In this section we have essentially reproduced Section 2 of [1b], in fact we have only changed the notations and the reason will be clear from the inductive proof we give of Proposition 5.3 (cf. Section 7).

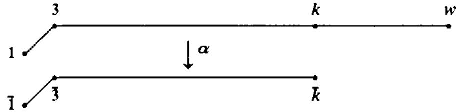
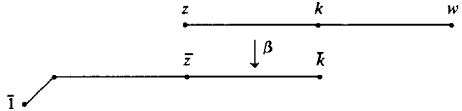
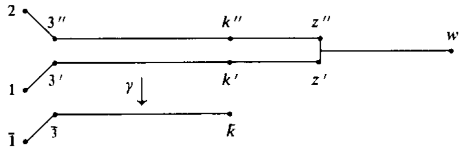
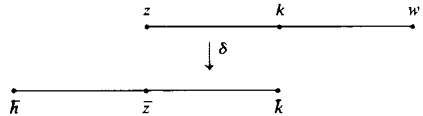
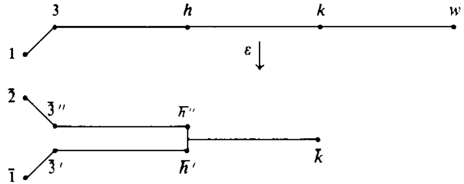
3. Rank Parameters for the Orbits

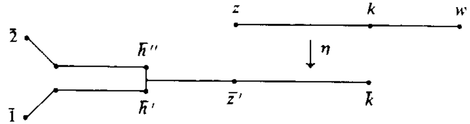
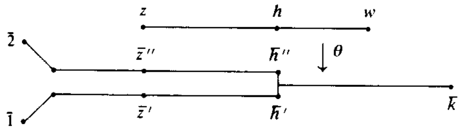
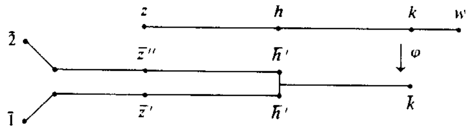
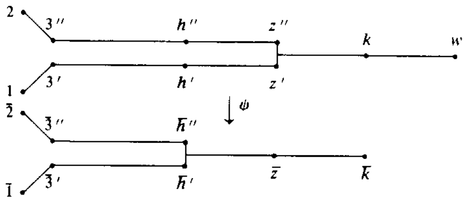
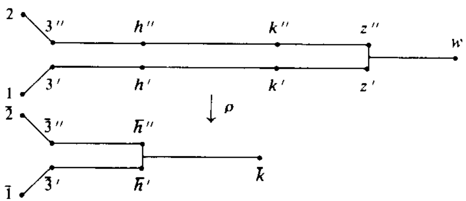
Let us first recall a parametrization of the set $\{O_A\}$, $A \in L_d$, of the orbits of G_d in L_d (cf. Section 1) which have been introduced in [1b, Section 1]. Again, for technical reasons, we are forced to change here the notations but the content is the same as in [1]; for this reason we will not give the proofs of the statements, and the reader will easily recuperate then from [1b].

Let

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}; A_3, A_4, ..., A_{m-1}\Big)$$

TABLE 2

	Graphic description	Independent generators	Dimension
$\mathrm{Hom}(E_{1w}, E_{1k})$ $k \leqslant w$		α: $1 \mapsto \bar{1}, 3 \mapsto \bar{3},\ldots, k \mapsto \bar{k}$	1
$\mathrm{Hom}(E_{2w}, E_{1k})$ $3 \leqslant z \leqslant k \leqslant w$		β: $z \mapsto \bar{z},\ldots, k \mapsto \bar{k}$	1
$\mathrm{Hom}(F_{zw}, E_{1k})$ $k \leqslant z$		γ: $1 \mapsto \bar{1}, 3' \mapsto \bar{3},\ldots, k' \mapsto \bar{k}$	1
$\mathrm{Hom}(E_{zw}, E_{hk})$ $3 \leqslant h \leqslant z \leqslant k \leqslant w$		δ: $z \mapsto \bar{z},\ldots, k \mapsto \bar{k}$	1
$\mathrm{Hom}(E_{1w}, F_{hk})$ $k \leqslant w$		ε: $1 \mapsto \bar{1}, 3 \mapsto \overline{3'},\ldots, h \mapsto \overline{h'}$; $h+1 \mapsto \frac{1}{2}\overline{(h+1)},\ldots, k \mapsto \frac{1}{2}\bar{k}$	1

$\mathrm{Hom}(E_{zw}, F_{hk})$ $0 \leqslant h < z \leqslant k \leqslant w$		$\eta: z \mapsto \bar{z}, \dots, k \mapsto \bar{k}$	1
$\mathrm{Hom}(E_{zw}, F_{hk})$ $3 \leqslant z \leqslant h \leqslant w \leqslant k$		$\theta: z \mapsto \bar{z}' - \bar{z}'', \dots, h \mapsto \bar{h}' - \bar{h}''$	1
$\mathrm{Hom}(E_{zw}, F_{hk})$ $3 \leqslant z \leqslant h < k \leqslant w$		$\varphi_1: z \mapsto \bar{z}', \dots, h \mapsto \bar{h}';$ $h+1 \mapsto \frac{1}{2}(\overline{h+1}), \dots, k \mapsto \frac{1}{2}\bar{k}$ $\varphi_2: z \mapsto \bar{z}'', \dots, h \mapsto \bar{h}'';$ $h+1 \mapsto \frac{1}{2}(\overline{h+1}), \dots, k \mapsto \frac{1}{2}\bar{k}$	2
$\mathrm{Hom}(F_{zw}, F_{hk})$ $0 \leqslant h \leqslant z < k \leqslant w$		$\psi: 1 \mapsto \bar{1}, 2 \mapsto \bar{2}, 3'' \mapsto \bar{3}'', 3' \mapsto \bar{3}', \dots, h'' \mapsto \bar{h}'',$ $h' \mapsto \bar{h}', (h+1)'' \mapsto \frac{1}{2}(\overline{h+1}), (h+1)' \mapsto \frac{1}{2}(\overline{h+1}), \dots,$ $z'' \mapsto \frac{1}{2}\bar{z}, z' \mapsto \frac{1}{2}\bar{z}, z+1 \mapsto \overline{z+1}, \dots, k \mapsto \bar{k}$	1
$\mathrm{Hom}(F_{zw}, F_{hk})$ $0 \leqslant h < k \leqslant z$		$\rho_1: 1 \mapsto \bar{1}, 3' \mapsto \bar{3}', \dots, h' \mapsto \bar{h}', (h+1)' \mapsto$ $\frac{1}{2}(\overline{h+1}), \dots, k' \mapsto \frac{1}{2}\bar{k}$ $\rho_2: 2 \mapsto \bar{2}, 3'' \mapsto \bar{3}'', \dots, h'' \mapsto \bar{h}''. (h+1)'' \mapsto$ $\frac{1}{2}(\overline{h+1}), \dots, k'' \mapsto \frac{1}{2}\bar{k}$	2

be a representation of $(D_m, \text{eq.})$ of dimension d:

$$\begin{tikzcd}
V_2 \arrow[dr, "A_2"] & & & & & & \\
& V_3 \arrow[r, "A_3"] & V_4 \arrow[r, "A_4"] & V_5 \cdots\cdots & V_{m-1} \arrow[r, "A_{m-1}"] & V_m \\
V_1 \arrow[ur, "A_1"'] & & & & & &
\end{tikzcd}$$

and let us denote by $A_0 : V_0 := V_1 \oplus V_2 \to V_3$ the linear map defined by $A_0(v_1, v_2) = A_1 v_1 + A_2 v_2$.

To the representation A we associate the following linear maps:

Definitions and Notations 3.1.

(1) $\varphi^A_{hk} = A_{k-1} \circ \cdots \circ A_{h+1} \circ A_h : V_h \to V_k,\ h < k,\ h \in \{1, 2, 3, \ldots, m\}$

(2) $\varphi^A_{0k} = A_{k-1} \circ \cdots \circ A_3 \circ A_0 : V_0 = V_1 \oplus V_2 \to V_k,\ k \in [0, m]$

(3) $\Psi^A_{hk} : V_1 \oplus V_2 \oplus V_h \to (V_h \oplus V_h \oplus V_h)/\Delta \oplus V_k,\ h < k,\ h \in [0, m]$

$$\Psi^A_{hk}(v_1, v_2, v_h) = (\overline{(\varphi^A_{1h} v_1, \varphi^A_{2h} v_2, v_h)}, \varphi^A_{hk} v_h)$$

where $\overline{(\varphi^A_{1h} v_1, \varphi^A_{2h} v_2, v_h)}$ is the class of $(\varphi^A_{1h} v_1, \varphi^A_{2h} v_2, v_h)$ modulo the diagonal Δ of $V_h \oplus V_h \oplus V_h$.

Definition of the Rank Parameters 3.2.

(1) $N^A_{hk} := \operatorname{rk} \varphi^A_{hk},\ h < k,\ h \in \{1, 2, 3, \ldots, m\}$

(2) $N^A_{0k} := \operatorname{rk} \varphi^A_{0k} = \operatorname{rk} \varphi^A_{1k} + \operatorname{rk} \varphi^A_{2k} - \dim(\operatorname{Im} \varphi^A_{1k} \cap \operatorname{Im} \varphi^A_{2k}),\ k \in [3, m]$

(3) $M^A_{hk} := \operatorname{rk} \psi^A_{hk} - \dim V_h$
$= \operatorname{rk} \varphi^A_{1k} + \operatorname{rk} \varphi^A_{2k} - \dim(\operatorname{Im} \varphi^A_{1h} \cap \operatorname{Im} \varphi^A_{2h} \cap \ker \varphi^A_{hk}),$
$h < k, h \in [3, m-1]$

(4) $N^A_{ii} = \dim V_i = d_i,\ i = 1, 2, \ldots, m,\ N^A_{00} = d_1 + d_2.$

We set $N^A = \{N^A_{hk}, M^A_{hk}\}$ and we call the non-negative integers of N^A the "rank parameters of the representation A." The reason is the following. Consider the decomposition of A into indecomposable factors (direct summands):

$$A = \bigoplus e^A_{pq} E_{pq} \oplus \bigoplus f^A_{rs} F_{rs}$$

where e^A_{pq} (resp. f^A_{rs}) denote the multiplicity of E_{pq} (resp. F_{rs}) in A. Clearly we can express the rank parameters of N^A in terms of the multiplicities e^A_{pq},

f^A_{rs}, as ranks are additive (and the dimensions too). Explicitly we have the linear system:

$$\begin{aligned}
N_{\tau\, s} &= (e_{\tau s} + e_{\tau\, s+1} + \cdots + e_{\tau m}) + (f_{0s} + f_{0\, s+1} + \cdots + f_{0m}) \\
&\quad + (f_{3s} + \cdots + f_{3m}) + \cdots + (f_{s\, s+1} + \cdots + f_{s\, m}) + \cdots + f_{m-1\, m}, \\
&\qquad \tau = 1, 2 \\
N_{r\, s} &= (e_{1s} + \cdots + e_{1m}) + (e_{2s} + \cdots + e_{2m}) + (e_{3s} + \cdots + e_{3m}) \\
&\quad + \cdots + (e_{rs} + \cdots + e_{rm}) + (f_{0\, s} + \cdots + f_{0\, m}) + (f_{3s} + \cdots + f_{3m}) \\
&\quad + \cdots + (f_{s-1\, s} + \cdots + f_{s-1\, m}) + 2(f_{s\, s+1} + \cdots + f_{sm}) + \cdots + 2f_{m-1\, m}, \\
&\qquad r \in [0, m],\ r \leqslant s \qquad (3.3) \\
M_{r\, s} &= (e_{1r} + \cdots + e_{1m}) + (e_{2r} + \cdots + e_{2m}) + (f_{0r} + \cdots + f_{0\, s-1} \\
&\quad + 2f_{0s} + \cdots + 2f_{0m}) + (f_{3r} + \cdots + f_{3\, s-1} + 2f_{3s} + \cdots + 2f_{3m}) \\
&\quad + \cdots + (f_{r-1\, r} + \cdots + f_{r-1\, s-1} + 2f_{r-1} + \cdots + 2f_{r-1\, m}) \\
&\quad + 2(f_{r\, r+1} + \cdots + f_{rm}) + \cdots + 2f_{m-1\, m}, \\
&\qquad r < s,\ r \in [3, m-1].
\end{aligned}$$

In (3.3) we mean that some summands vanish for special values of the indices. For example, we have:

$$\begin{aligned}
N_{0\, s} &= (e_{1s} + \cdots + e_{1m}) + (e_{2s} + \cdots + e_{2m}) + (f_{0s} + \cdots + f_{0m}) \\
&\quad + (f_{3s} + \cdots + f_{3m}) + \cdots + (f_{s-1\, s} + \cdots + f_{s-1\, m}) \\
&\quad + 2(f_{s\, s+1} + \cdots + f_{sm}) + \cdots + 2f_{m-1\, m}.
\end{aligned}$$

In [1] we have proved that (3.3) is invertible over the integers, therefore the sets N^A, $A \in L_d$, parametrize the orbits. Moreover, we have the following:

PROPOSITION 3.4 (Cf. 1.5 of [1b]). *Let $A, B \in L_d(D_n, \text{eq.})$ be such that $O_B \subset \bar{O}_A$ and $O_B \neq O_A$, then*

$$N^B_{rs} \leqslant N^A_{rs}, \qquad M^B_{rs} \leqslant M^A_{rs}$$

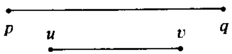
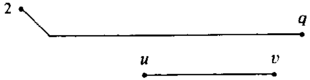
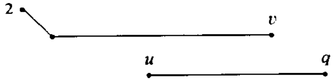
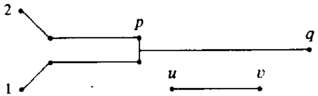
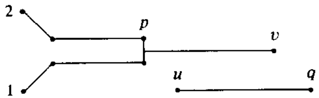
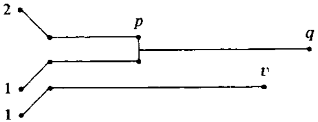
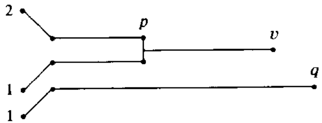
for all pairs (r, s), $r \neq s$, and the strict inequality holds for at least a pair of indices.

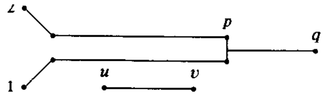
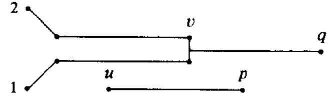
4. ELEMENTARY DEGENERATIONS

We introduce here some operations $\mathscr{D}$ on the indecomposable representations of $\mathbb{Q}_m = (D_m, \text{eq.})$ called "elementary degenerations" (cf. 4.3) which

TABLE 3

Elementary Degenerations in D_m

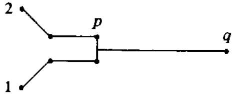
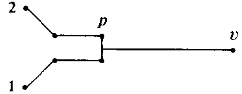
$\mathscr{D}$	Column 1 **A**	Column 2 $\mathscr{D}\mathbf{A} = \mathbf{C} = P \oplus Q$	Column 3 $(a, b) \in \mathrm{ob}(\mathscr{D}) = \bigcup_{\rho=\alpha,\beta,\gamma} H_\rho \times K_\rho$	
I	 $3 \leqslant p < u \leqslant v < q$		$(\alpha)\quad a \in [p, u-1] = H_\alpha,$	$b \in [v+1, q] = K_\alpha$
II	 $3 \leqslant u \leqslant v < q$		$(\alpha)\quad a \in [0, u-1] \cup \{2\} = H_\alpha,$ $(\beta)\quad a \in [v+1, q] = H_\beta,$	$b \in [v+1, q] = K_\alpha$ $b \in [v+2, m] = K_\beta$
III	 $3 \leqslant u \leqslant v < q$		$(\alpha)\quad a \in [0, u-1] \cup \{1, 2\} = H_\alpha,$ $(\beta)\quad \begin{cases} a \in [p+1, v] = H'_\beta \\ a \in [v+1, q] = H''_\beta, \end{cases}$ $(\gamma)\quad a \in [v+1, q-1] = H_\gamma$	$b \in [v+1, q] = K_\alpha$ $b \in [v+1, q] = K'_\beta$ $b \in [q+1, m] = K''_\beta$ $b \in [v+2, q] = K_\gamma$
IV	 $0 \leqslant p < v < q$		$(\alpha)\quad a \in \{2\} = H_\alpha,$ $(\beta)\quad a \in [p+1, q-1] = H_\beta,$	$b \in [v+1, q] = K_\alpha$ $b \in [v+1, q] = K_\beta$

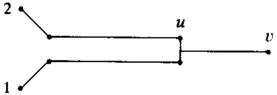
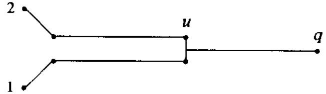
(α) $a \in [0, u-1] = H_\alpha$, $b \in [v+1, p] = K_\alpha$
(β) $a \in [v+1, p] = H_\beta$, $b \in [q+1, m] = K_\beta$

$3 \leqslant u \leqslant v < p < q$

VI

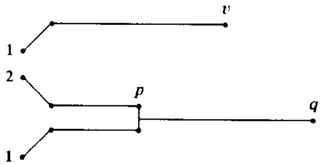
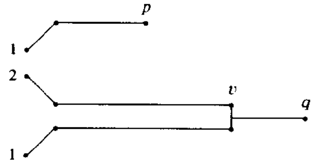
 

(β) $a \in [p+1, u] = H_\beta$, $b \in [v+1, q] = K_\beta$

$0 \leqslant p < u < v < q$

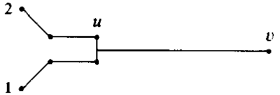
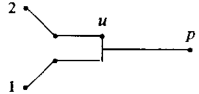
VII

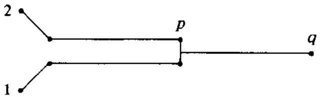
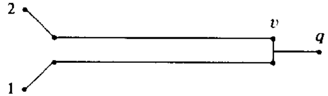
(α) $a \in \{1\} = H_\alpha$, $b \in [p+1, v] = K_\alpha$
(β) $a \in [p+1, v] = H_\beta$, $b \in [p+2, q] = K_\beta$

$0 < p < v < q$

VIII

(α) $a \in \{1, 2\} = H_\alpha$, $b \in [p+1, v] = K_\alpha$
(β) $\begin{cases} a \in [u+1, p] = H'_\beta, & b \in [p+1, v] = K'_\beta \\ a \in [p+1, v] = H''_\beta, & b \in [p+1, q] = K''_\beta \end{cases}$
(γ) $a \in [p+1, v-1] = H_\gamma$ $b \in [p+2, v] = K_\gamma$

$0 \leqslant u < p < v < q$

Table continued

TABLE 3 (*continued*)

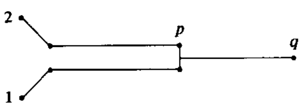
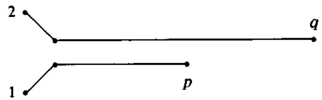
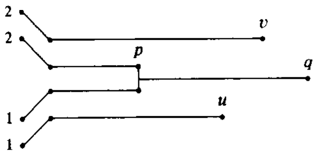
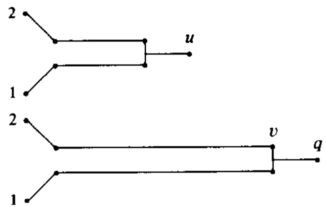
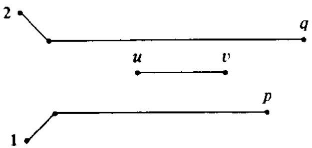
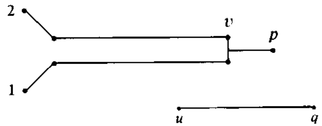
$\mathscr{D}$	Column 1 **A**	Column 2 $\mathscr{D}\mathbf{A} = \mathbf{C} = P \oplus Q$	Column 3 $(a, b) \in \mathrm{ob}(\mathscr{D}) = \bigcup_{\rho=\alpha,\beta,\gamma} H_\rho \times K_\rho$	
IX	 $0 < p < q$		(α) $a \in \{2\} = H_\alpha$, (β) $a \in [p+1, q-1] = H_\beta$,	$b \in [p+1, q] = K_\alpha$ $b \in [p+2, q] = K_\beta$
X	 $0 \leqslant p < u \leqslant v < q$		(α) $a \in \{2\} = H_\alpha$, (β) $a \in [p+1, v] = H_\beta$,	$b \in [u+1, v] = K_\alpha$ $b \in [u+1, q] = K_\beta$
XI	 $3 \leqslant u \leqslant v < p \leqslant q$		(α) $\begin{cases} a \in \{2\} = H'_\alpha, \\ a \in [0, u-1] = H''_\alpha, \end{cases}$ (β) $a \in [v+1, q] = H_\beta$,	$b \in [p+1, q] = K'_\alpha$ $b \in [v+1, q] = K''_\alpha$ $b \in [p+1, m] = K_\beta$

TABLE 3′

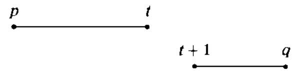
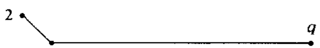
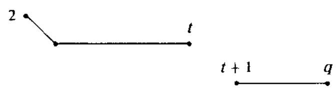
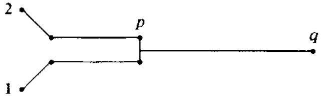
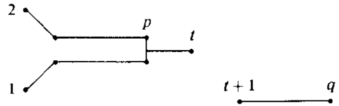
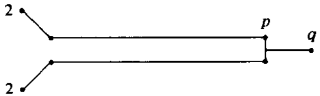
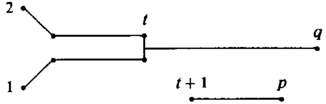
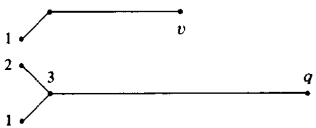
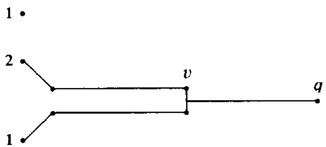
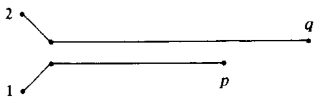
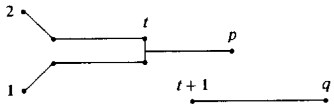
$\mathscr{D}$	Column 1 **A**	Column 2 $\mathscr{D}\mathbf{A} = \mathbf{C} = P \oplus Q$	Column 3 $(a, b) \in \mathrm{ob}(\mathscr{D})$	
I′	 $p \leqslant t < q$		(α) $a \in [p, t]$,	$b \in [t+1, q]$
II′	 $2 \leqslant t < q$		(α) $a \in [0, t] \cup \{2\}$, (β) $a \in [t+1, q]$,	$b \in [t+1, q]$ $b \in [t+2, m]$
III′	 $0 \leqslant p < t < q$		(α) $a \in [0, t] \cup \{1, 2\}$, (β) $\begin{cases} a \in [p+1, t], \\ a \in [t+1, q], \end{cases}$ (γ) $a \in [t+1, q-1]$,	$b \in [t+1, q]$ $b \in [t+1, q]$ $b \in [q+1, m]$ $b \in [t+2, q]$

Table continued

TABLE 3′ (*continued*)

$\mathscr{D}$	Column 1 **A**	Column 2 $\mathscr{D}\mathbf{A} = \mathbf{C} = P \oplus Q$	Column 3 $(a, b) \in \mathrm{ob}(\mathscr{D})$	
V′	 $0 \leqslant t < p$		(α) $a \in [0, t]$, (β) $a \in [t+1, p]$,	$b \in [t+1, p]$ $b \in [q+1, m]$
VII′	 $3 \leqslant v < q$		(α) $a \in \{1\}$, (β) $a \in [3, v]$,	$b \in [3, v]$ $b \in [4, q]$
IX′	 $3 \leqslant q$		(α) $a \in \{1\}$, (β) $a \in [3, q-1]$,	$b \in [3, q]$ $b \in [q+1, m]$
XI′			(α) $\begin{cases} a \in \{2\}, \\ a \in [0, t], \end{cases}$ (β) $a \in [t+1, q]$,	$b \in [p+1, q]$ $b \in [t+1, q]$ $b \in [p+1, m]$

will generate a preorder relation in the set of orbits of given dimension. We collect them in Tables 3 and 3'. In column 1 we draw (for convenience) the indecomposables on which we perform the operation $\mathscr{D}$ (we denote by $\mathbf{A}$ their direct sum), and in column 2 we draw the representation $\mathbf{C} = \mathscr{D}\mathbf{A}$ which is the result of the operation.

From Corollary 4.3 we will have that $O_{\mathbf{C}} \subset \bar{O}_{\mathbf{A}}$, therefore, passing from $\mathbf{A}$ to $\mathbf{C}$ some of the rank parameters decrease their value (cf. Proposition 3.4). We compute them for each pair $\mathbf{A}, \mathbf{C} = \mathscr{D}\mathbf{A}$ using Definitions 3.1 and the interpretation of the graphic description of the indecomposables (cf. Section 2).

We have the following situation:

$$\begin{aligned} &(\alpha) \quad N^{\mathbf{A}}_{ab} = N^{\mathbf{C}}_{ab} - 1 \qquad &&\text{for } a \in H_\alpha,\ b \in K_\alpha \\ &(\beta) \quad M^{\mathbf{A}}_{ab} = M^{\mathbf{C}}_{ab} - 1 \qquad &&\text{for } a \in H_\beta,\ b \in K_\beta \\ &(\gamma) \quad M^{\mathbf{A}}_{ab} = M^{\mathbf{C}}_{ab} - 2 \qquad &&\text{for } a \in H_\gamma,\ b \in K_\gamma \end{aligned} \tag{4.1}$$

where H_τ's, $\tau = \alpha, \beta, \gamma$ (resp. the K_τ's) are subsets of the set $\{0, 1, 2, 3, ..., m\}$; all the remaining rank parameters are unchanged.

In column 3 we list explicitly the sets H_τ, K_τ, $\tau = \alpha, \beta, \gamma$. We call (a, b) a pair of type ρ, $\rho = \alpha, \beta, \gamma$, if $(a, b) \in H_\rho \times K_\rho$ and we denote by

$$\mathrm{ob}(\mathscr{D}) = \bigcup_{\rho = \alpha, \beta, \gamma} H_\rho \times K_\rho .$$

The pairs of indices $(a, b) \in \mathrm{ob}(\mathscr{D})$ are called obstruction indices for the degeneration $\mathscr{D}$, and reason will be clear from Remark 5.4.

Remarks. We want to compare Tables 3' and 3. Tha degenerations I', II', III', V', XI' can be thought of as special cases resp. of I, II, III, V, XI; it is enough to assume there $u = t + 1$, $v = t$ and read conventionally the indecomposable $E_{uv} = E_{t+1\,t}$ of column 1 as the zero representation at the vertex t. Note that the obstruction indices of column 3 are coherent with such convention. VII' and IX' can be considered a special case of VII and IX for $p = 0$; in this case in column 2 the indecomposable E_{1p} is not defined, and conventionally we read it as the irreducible representation E_{11}. Again the obstruction induces of column 3 are consistent with this convention.

Let us consider the graph automorphism $\sigma = (1, 2)$ of $(D_m$, eq.) induced by the permutation of the vertices 1 and 2, and consider the representations $\mathbf{A}$ defined in column 1 of Table 3 (resp. Table 3'). In the cases II, IV, VII, X, XI we have that $\sigma\mathbf{A} \neq \mathbf{A}$. It follows that we have also to consider the operation which associates to the representation $\sigma\mathbf{A}$ the representation $\sigma\mathbf{C}$. In case IX we have $\sigma\mathbf{A} = \mathbf{A}$ but $\sigma\mathbf{C} \neq \mathbf{C}$, therefore we have also to consider the operation that to $\mathbf{A}$ associates $\sigma\mathbf{C}$.

Note that in Table 3 column 3 some of the sets $H_\rho \times K_\rho$ may lose sense for special values of the indices. For example, for the degeneration III, if $q = m$, the set $K''_\beta = [q+1, m]$ is meaningless. In all these cases we set $H_\rho \times K_\rho = \varnothing$.

From now on we will use these conventions, and we will explicitly discuss only the operations listed in Table 3; but the statements we make hold for the operations listed in Table 3′ and for the new operations defined using the graph automorphism σ.

Note that $\mathbf{C}$ always contains two indecomposable factors, therefore we have written $\mathbf{C} = P \oplus Q$, and for each case (I,..., XI) we mean that Q is the second indecomposable of column 2.

PROPOSITION 4.2. *Let* $\mathbf{A}, \mathbf{C} = P \oplus Q$ *be a pair of representations listed in Table* 3, *cases I,..., XI (resp. Table* 3′*). Then*:

$$\mathbf{A} \in \mathrm{Ext}_K(P, Q).$$

Proof. We have to show case by case that there exists an injective map $j: Q \hookrightarrow \mathbf{A}$ such that $P \simeq \mathbf{A}/jQ$.

To specify the morphism j we use the generators listed in Table 2 of Section 2, as follows:

Case I. $\delta \in \mathrm{Hom}_K(E_{uq}, E_{pq})$, $\tilde{\delta} \in \mathrm{Hom}_K(E_{uq}, E_{uv})$; $j = \delta - \tilde{\delta}$.

Case II. $\beta \in \mathrm{Hom}_K(E_{uq}, E_{2q})$, $\delta \in \mathrm{Hom}_K(E_{uq}, E_{uv})$; $j = \beta - \delta$.

Case III. If $p < u$, $\eta \in \mathrm{Hom}_K(E_{uq}, F_{pq})$, $\delta \in \mathrm{Hom}_K(E_{uq}, E_{uv})$; $j = \eta - \delta$. If $3 \leqslant u \leqslant p$, $\varphi = \varphi_1 - \varphi_2 \in \mathrm{Hom}_K(E_{uq}, F_{pq})$, $\delta \in \mathrm{Hom}_K(E_{uq}, E_{uv})$; $j = \varphi - \delta$.

Case IV. $\varepsilon \in \mathrm{Hom}_K(E_{1q}, F_{pq})$, $\alpha \in \mathrm{Hom}_K(E_{1q}, E_{1v})$; $j = \varepsilon - \alpha$.

Case V. $\theta \in \mathrm{Hom}_K(E_{up}, F_{pq})$, $\delta \in \mathrm{Hom}_K(E_{up}, E_{uv})$; $j = \theta - \delta$.

Case VI. $\psi \in \mathrm{Hom}_K(F_{uq}, F_{uv})$, $\psi \in \mathrm{Hom}_K(F_{uq}, F_{pq})$; $j = \psi - \tilde{\psi}$.

Case VII. $\psi \in \mathrm{Hom}_K(F_{vq}, F_{pq})$, $\gamma \in \mathrm{Hom}_K(F_{vq}, E_{1v})$; $j = \psi - \gamma$.

Case VIII. $\psi \in \mathrm{Hom}_K(F_{vq}, F_{pq})$, $\tilde{\psi} \in \mathrm{Hom}_K(F_{vq}, F_{uv})$; $j = \psi - \tilde{\psi}$.

Case IX. $\varepsilon \in \mathrm{Hom}_K(E_{1q}, F_{pq})$; $j = \varepsilon$.

Case X. $\gamma \in \mathrm{Hom}_K(F_{vq}, E_{2v})$, $\tilde{\gamma} \in \mathrm{Hom}_K(F_{vq}, E_{1q})$, $\psi \in \mathrm{Hom}_K(F_{vq}, F_{pq})$; $j = \gamma + \tilde{\gamma} + \psi$.

Case XI. $\beta \in \mathrm{Hom}_K(E_{uq}, E_{2q})$, $\tilde{\beta} \in \mathrm{Hom}_K(E_{uq}, E_{1p})$, $\delta \in \mathrm{Hom}_K(E_{uq}, E_{uv})$; $j = \beta + \tilde{\beta} + \delta$.

It is easy to check that j is an injective morphism of modules and that $P \simeq \mathbf{A}/jQ$.

COROLLARY 4.3. *Let* $\mathbf{A}$, $\mathbf{C}$ *be a pair of representations listed in Table* 3 (*resp. Table* 3′), *then* $O_{\mathbf{C}} \subset \bar{O}_{\mathbf{A}}$.

Proof. Clearly $[\mathbf{A}] \neq [\mathbf{C}]$ and $\dim \mathbf{C} = \dim \mathbf{A}$. Moreover, from Proposition 4.2 we have that $0 \to Q \to \mathbf{A} \to P \to 0$ is an exact sequence and $\mathbf{C} = P \oplus Q$ (cf. [3]).

We consider now the variety L_d and the action of the group G_d defined in Section 1.

DEFINITION 4.4. Let $A \in L_d$, we say that we perform on A an elementary degeneration $\mathscr{D}$ if $A = \mathbf{A} \oplus M$, where $\mathbf{A}$ is one of the representations listed in Table 3 (resp. Table 3′) column 1 and the result of $\mathscr{D}$ is the representation $C = \mathbf{C} \oplus M$, where $\mathbf{C}$ is the representation corresponding to $\mathbf{A}$ in column 2:

$$\mathscr{D} : A = \mathbf{A} \oplus M \mapsto C = \mathbf{C} \oplus M.$$

Clearly we have $O_C \subseteq \bar{O}_A$, and the equalities (4.1) hold for A and C, as ranks are additive.

The elementary degenerations generate a preorder relation in the set of orbits of the action of G_d in L_d. We will see in Section 5 that this is in fact an ordering which we call "the combinatorial ordering" and denote it by $\leqslant_c$. The definition is the following.

DEFINITION 4.5. Let $A, B \in L_d$, we say that $O_B \leqslant_c O_A$ if and only if B is obtained from A performing a finite number of elementary degenerations.

Let us denote by $\leqslant_g$ the geometrical ordering of the orbits in L_d given by:

DEFINITION 4.6. *Let* A, $B \in L_d$, *we say that* $O_B \leqslant_g O_A$ if and only if $O_B \subseteq \bar{O}_A$ (i.e., O_B is a degeneration of O_A).

As a consequence of 4.3 we have:

PROPOSITION 4.7. *For* $A, B \in L_d$, *if* $O_B \leqslant_c O_A$ *then* $O_B \leqslant_g O_A$.

5. THE MAIN THEOREM: STRATEGY AND SKETCH OF THE PROOF

Let us consider the variety $L_d = L_d(D_m, \text{eq.})$ defined in Section 1, on which acts the group G_d.

If the set of the orbits of this action we define a "rank ordering," denoted by $\leqslant_r$, as follows.

DEFINITION 5.1. Let $A, B \in L_d$, we say that $O_B \leqslant_r O_A$ if and only if

$$\begin{aligned} &N^B_{rs} \leqslant N^A_{rs}, \qquad r<s, \quad r \in [0, m-1) \cup \{1,2\} \\ &M^B_{rs} \leqslant M^A_{rs}, \qquad r<s, \quad r \in [3, m-1]. \end{aligned} \tag{5.1}$$

Proposition 3.4 says that if $O_B \leqslant_g O_A$ then $O_B \leqslant_r O_A$. The theorem we want to prove is the following:

THEOREM 5.2. *The three orderings $\leqslant_c, \leqslant_g, \leqslant_r$ coincide.*

We only need to compare the orderings $\leqslant_c$ and $\leqslant_r$ (cf. 4.6), and the strategy we will use is described by the following Proposition 5.3. To simplify the notations, from now on we will write $B \leqslant A$ instead of $O_B \leqslant_r O_A$.

PROPOSITION 5.3. *Let $A, B \in L_d$ be such that $B < A$. Then there exists a $C \in L_d$ obtained from A via an elementary degeneration and such that $B \leqslant C < A$.*

Remark 5.4. Let A, $B \in L_d$ and $B < A$; if we perform an elementary degeneration $\mathscr{D}$ on A, i.e., $A = \mathbf{A} \oplus M$, and $\mathbf{A}$ listed in Table 3 (resp. Table 3′) column 1 of Section 4, we do get $C = \mathbf{C} \oplus M$ ($\mathbf{C}$ defined in column 2) with $C < A$, but in order to have $B \leqslant C = \mathscr{D}A$ the rank parameters of the given representations A and B must satisfy the following inequalities:

$$\begin{aligned} &(\alpha) \quad N^B_{ab} < N^A_{ab} \qquad \text{for every pair } (a,b) \in H_\alpha \times K_\alpha \\ &(\beta) \quad M^B_{ab} < M^A_{ab} \qquad \text{for every pair } (a,b) \in H_\beta \times K_\beta \\ &(\gamma) \quad M^B_{ab} < M^A_{ab} - 1 \quad \text{for every pair } (a,b) \in H_\gamma \times K_\gamma, \end{aligned} \tag{5.5}$$

$H_\tau \times K_\tau$, $\tau = \alpha, \beta, \gamma$ defined in Table 3, column 3, i.e., the obstruction indices of $\mathscr{D}$. Therefore the pairs $(a,b) \in \mathrm{ob}(\mathscr{D})$ are regarded as an obstruction to get

$$B \leqslant \mathscr{D}A = C < A. \quad \blacksquare$$

The statements of Theorem 5.2 and Proposition 5.3 have been proved in [2] for the representations of the Dynkin diagram A_m arbitrarily oriented. We will need here this results.

Let $Q_{m-1} = (D_{m-1}, \text{eq.})$ be the quiver obtained from $Q_m = (D_m, \text{eq.})$ erasing the last vertex m. To any representation

$$A = \left(\begin{matrix} A_2 \\ A_1 \end{matrix} ; A_3, \ldots, A_{m-2}, A_{m-1} \right)$$

of Q_m corresponds the representation

$$A' = \begin{pmatrix} A_2 \\ A_1 \end{pmatrix}; A_3, \dots, A_{m-2}\Big)$$

of Q_{m-1}. In particular to the indecomposable representation E_{pq}, $p \leqslant q \leqslant m-1$, $p = 1, 2, \dots, m-1$, or to F_{pq}, $0 \leqslant p < q \leqslant m-1$, corresponds the indecomposable representation E_{pq} or F_{pq} of Q_{m-1}; to the indecomposable E_{pm}, $p = 1, 2, \dots, m-1$, of Q_m corresponds the indecomposable $E_{p\, m-1}$ of Q_{m-1}; to the indecomposable F_{pm}, $p \in [0, m-1]$, of Q_m corresponds the indecomposable $F_{p\, m-1}$ if $p < m-1$; corresponds the pair of factors $E_{1\, m-1} \oplus E_{2\, m-1}$ if $p = m-1$.

Let A be any representation of Q_m and A' the corresponding one in Q_{m-1}. We use their decomposition into indecomposable factors:

$$A = \sum e_{pq}^A E_{pq} + \sum f_{pq}^A F_{pq}, \qquad A' = \sum e_{rs}^{A'} E_{rs} + \sum f_{rs}^{A'} F_{rs}$$

5then for the rank parameters and for the multiplicities we have the following relations:

$$\begin{aligned} N_{ab}^{A'} &= N_{ab}^A, \qquad a \leqslant b \leqslant m-1 \\ M_{ab}^{A'} &= M_{ab}^A, \qquad a < b \leqslant m-1 \end{aligned} \tag{5.6}$$

$$\begin{aligned} e_{rs}^{A'} &= e_{rs}^A, \qquad r \leqslant s < m-1 \\ e_{r\, m-1}^{A'} &= e_{r\, m-1}^A + e_{r\, m}^A, \qquad r \geqslant 3 \\ e_{\tau\, m-1}^{A'} &= e_{\tau\, m-1}^A + e_{\tau\, m}^A + f_{m-1\, m}^A, \qquad \tau = 1, 2 \\ f_{rs}^{A'} &= f_{rs}^A, \qquad 0 \leqslant r < s < m-1 \\ f_{r\, m-1}^{A'} &= f_{r\, m-1}^A + f_{r\, m}^A. \end{aligned} \tag{5.7}$$

Let $A, B \in L_d(D_m, \text{eq.})$ such that $B < A$, and let A', B' be the corresponding representations in $\mathbb{Q}_{m-1}$. From (5.6) it follows that

$$B' \leqslant A'.$$

Assume we know that E_{rs} (resp. F_{rs}) is a factor in A', i.e., $e_{rs}^{A'} > 0$ (resp. $f_{rs}^{A'} > 0$), then to the factor E_{rs} (resp. F_{rs}) in A' we can associate a factor in A (not necessarily the same factor and not necessarily unique), and we call such a factor in A "a lifting of E_{rs} (resp. F_{rs}) from A' to A." To do this we use the relations (5.7):

—If $e_{rs}^{A'} > 0$, $s < m-1$, then $e_{rs}^A > 0$, i.e., E_{rs} is a factor in A, and the lifting of E_{rs} from A' to A is E_{rs}.

—If $e^{A'}_{r\,m-1} > 0$, $r \geqslant 3$, then $e^{A}_{r\,m-1} + e^{A}_{r\,m} > 0$ and at least one of the two multiplicities is positive, it follows that at least one of the indecomposable $E_{r\,m-1}$, $E_{r\,m}$ is a factor in A. In this case a lifting of $E_{r\,m-1}$ from A' to A is either $E_{r\,m-1}$ if $e^{A}_{r\,m-1} > 0$, or $E_{t\,m}$ if $e^{A}_{r\,m} > 0$.

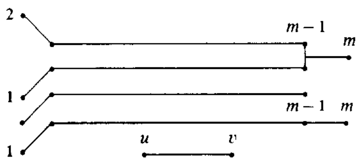
—If $e^{A'}_{\tau\,m-1} > 0$, $\tau = 1, 2$, then at least one of the indecomposable $E_{\tau\,m-1}$, $E_{\tau\,m}$, $F_{m-1\,m}$ is a factor in A, and a lifting of $E_{\tau\,m-1}$ from A' to A is any one of the previous factors of A.

—If $f^{A'}_{rs} > 0$, $r < s < m-1$, then F_{rs} is the lifting from A' to A of the same indecomposable.

—If $f^{A'}_{r\,m-1} > 0$ then at least one of the indecomposables $F_{r\,m-1}$, $F_{r\,m}$ is a factor in A and a lifting of $F_{r\,m-1}$ from A' to A is any one of the previous factors in A.

Assume now that we can perform on A' a degeneration $\mathscr{D}'$, i.e., $A' = \mathbf{A}' \oplus M'$, $\mathbf{A}'$ one of the representations listed in column 1 of Table 3 (cf. Section 4, where we read Table 3 for the representations of Q_{m-1}). If we lift all the factors of $\mathbf{A}'$ from A' we get a representation $\mathbf{A}$ of Q_m (not necessarily unique!) such that $A = \mathbf{A} \oplus M$ and it is easy to check that $\mathbf{A}$ is again one of the representations listed in Table 3, column 1. It follows that we can perform on A a degeneration $\mathscr{D}$ (not necessarily unique, and not necessarily of the same type of $\mathscr{D}'$). We call such a $\mathscr{D}$"a lifting of the degeneration $\mathscr{D}'$ from A' to A."

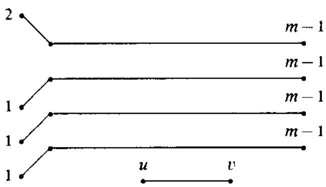
Example. Let $A = F_{m-1\,m} + E_{1\,m} + E_{1\,m-1} + E_{u\,v}$



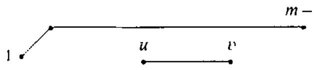
$3 \leqslant u < v < m-1$

then we have

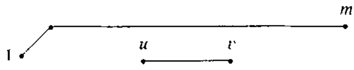
$$A' = E_{2\,m-1} + 3E_{1\,m-1} + E_{u\,v}$$



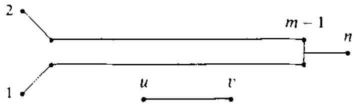
Assume we perform on A' a degeneration $\mathscr{D}'$ of type II, i.e., $A' = \mathbf{A}' \oplus M'$, $\mathbf{A}' = E_{1\ m-1} \oplus E_{u\ v}$, then the possible liftings of $\mathscr{D}'$ from A' to A are:



Type II



Type II



Type V

For the proof of Proposition 5.3, we will need the following:

DEFINITION 5.8. Let $A, B \in L_d(D_m, \text{eq.})$ such that $B < A$ and A', B' the corresponding representations in Q_{m-1}. Assume also that we can perform on A' a degeneration $\mathscr{D}'$ and on A a degeneration $\mathscr{D}^*$. Then we will say that $\mathscr{D}^*$ is "trivial will respect to $\mathscr{D}'$" if:

(1) $B' \leqslant \mathscr{D}'A' < A'$,

(2) $\mathrm{ob}(\mathscr{D}^*) \subset \mathrm{ob}(\mathscr{D}')$.

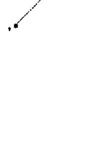
Note that if $\mathscr{D}^*$ is trivial with respect to $\mathscr{D}'$ then we have

$$B \leqslant \mathscr{D}^*A < A.$$

Sketch of the Proof of 5.3

Let $A, B \in L_d$ such that $B < A$, it follows that $B' \leqslant A'$. The proof of 5.3 will be done in two steps.

Step 1. If $B' < A'$ we proceed by induction on the number m of the vertices of the graph, and assume Proposition 5.3 for the pair A', B'. The initial case is for $m = 3$, where the oriented graph is



i.e., a Dynkin diagram of type A_l for $l = 3$ (cf. [2b]). Therefore by induction we know that there exists a degeneration $\mathscr{D}'$ on A' such that

$$B' \leqslant \mathscr{D}A' < A'$$

i.e., the inequalities (5.5) hold for A' and B' (which are representations of $(D_{m-1}, \text{eq.}) = Q_{m-1}$).

The first assumption we must make is the following:

(*) No degeneration $\mathscr{D}^*$ (on A) is trivial with respect to $\mathscr{D}'$.

Otherwise the representation C we are looking for is trivially found, namely, $C = \mathscr{D}^*A$ (cf. Definition 5.8).

We know that the degeneration $\mathscr{D}'$ and A' can be lifted to a degeneration $\mathscr{D}$ on A. As a consequence of (*) we only have to consider, for any $\mathscr{D}'$ (of type I, II,..., XI), all the possible liftings $\mathscr{D}$ which are non-trivial (with respect to $\mathscr{D}'$). For each such a pair $\mathscr{D}'$, $\mathscr{D}$, we compare $\text{ob}(\mathscr{D})$ with $\text{ob}(\mathscr{D}') = \bigcup_{\tau=\alpha,\beta,\gamma} H_\tau \times K_\tau$. The fact that $\mathscr{D}$ is non-trivial will imply that now obstruction indices arise and $\text{ob}(\mathscr{D}) - \text{ob}(\mathscr{D}') = \{(h, m)\colon h \in H_\tau$ for some τ, $\tau = \alpha, \beta, \gamma\}$.

The second assumption we must make is the following:

(**) At least one of the following equalities must hold for a pair $(h, m) \in \text{ob}(\mathscr{D}) - \text{ob}(\mathscr{D}')$.

(1) $N^A_{hm} = N^B_{hm}$, $\quad h \in H_\alpha$

(2) $M^A_{hm} = M^B_{hm}$, $\quad h \in H_\beta$ or $h \in H_\gamma$

(3) $M^A_{hm} = M^B_{hm} + 1$, $\quad h \in H_\gamma$.

Otherwise the representation C we are looking for is trivially found, namely, $C = \mathscr{D}A$.

The assumption (**) will allow us to deduce that it is not empty a set Λ of factors of A, and choosing factors in Λ in a suitable way, we will define a degeneration $\tilde{\mathscr{D}}$ on A (cf. 7.I, 7.II,..., 7.XI) such that

$$(\square) \qquad B \leqslant \tilde{\mathscr{D}}A < A.$$

To prove this assertion we have to compare once more $\text{ob}(\mathscr{D}')$ and $\text{ob}(\tilde{\mathscr{D}})$ and prove the inequalities (5.5) only for the pairs $(r, s) \in \text{ob}(\tilde{\mathscr{D}}) - \text{ob}(\mathscr{D}')$, since for the pairs $(r, s) \in \text{ob}(\mathscr{D}')$ we already know that (5.5) hold.

The Proposition 5.3 is now proved under the assumption $B' < A'$.

Step 2. We may now assume that $B' = A'$; it follows that

$$N^A_{rs} = N^B_{rs} \qquad \text{for } r \in [0, m-1] \cup \{1, 2\},\ r \leqslant s < m$$

$$M^A_{rs} = M^B_{rs} \qquad \text{for } r \in [3, m-1],\ r < s < m$$

and $B < A$ implies that the strict inequality must hold for at least a rank parameter N_{rm} or M_{rm}.

In this case we will determine directly a representation $C = \mathscr{D}A$ satisfying the required property.

Both for steps 1 and 2 we will also assume

$$(***) \qquad \text{if } e^A_{pq} > 0 \text{ (resp. } f^A_{pq} > 0\text{), then } e^B_{pq} = 0 \text{ (resp. } f^B_{pq} = 0\text{).}$$

In fact it is clear that we may assume A and B such that they have no common indecomposable factors.

To prove step 1 we will need some identities. We collect them in two lemmas in the following section.

6. Lemmas

The identities of the following lemmas can be easily checked by direct computation, using the expressions (3.3).

Lemma 6.1. *For any representation of $Q_m = (D_m, eq.)$ we have the following identities*:

(1) $N_{h\ m-1} - N_{h\ m} = e_{1\ m-1} + e_{2\ m-1} + e_{3\ m-1} + \cdots + e_{h\ m-1} + f_{0\ m-1} + f_{3\ m-1} + \cdots + f_{m-2\ m-1} + f_{m-1\ m}$; $h \in [3, m-2]$.

(2) $N_{\tau\ m} + N_{0\ h} - M_{h\ m} = e_{\tau\ m} + f_{h\ m} + f_{h+1\ m} + \cdots + f_{m-1\ m}$; $\tau = 1, 2$, $h \in [3, m-1]$.

(3) $N_{\tau\ m-1} - N_{\tau\ m} = e_{\tau\ m-1} + f_{0\ m-1} + f_{3\ m+1} + \cdots + f_{m-2\ m-1}$; $\tau = 1, 2$.

(4) $N_{0\ m} - N_{1\ m} = e_{2\ m}$.

(5) $M_{h\ m-1} - M_{h\ m} = f_{0\ m-1} + f_{3\ m-1} + \cdots + f_{h-1\ m-1}$; $h \in [3, m-1]$, *where, if* $h = m-1$, *we substitute* $M_{h\ m-1}$ *with* $N_{1\ m-1} + N_{2\ m-1}$.

(6) $N_{1\ m} - N_{2\ m} = e_{1\ m} - e_{2\ m}$.

Lemma 6.2. *For any representation of $Q_m = (D_m, eq.)$ we have the following identities*:

(1) $N_{r\ s} = N_{r\ m} + N_{h\ s} - N_{h\ m} - [(e_{r+1\ s} + \cdots + e_{r+1\ m-1}) + \cdots + (e_{h\ s} + \cdots + e_{h\ m-1})]$; $r < h < s < m$.

(2) $N_{2\ s} = N_{2\ m} + N_{h\ s} - N_{h\ m} - [(e_{1\ s} + \cdots + e_{1\ m-1}) + (e_{3\ s} + \cdots + e_{3\ m-1}) + \cdots + (e_{h\ s} + \cdots + e_{h\ m-1}) + (f_{s\ s+1} + \cdots + f_{s\ m}) + \cdots + (f_{m-2\ m-1} + f_{m-2\ m}) + f_{m-1\ m}]$; $0 \leqslant h < s < m$, *where if* $h = 0$ *we mean that* $(e_{3\ s} + \cdots + e_{3\ m-1}) + \cdots + (e_{h\ s} + \cdots + e_{h\ m-1})$ *does not appear.*

(3) $M_{r\,s} = N_{h\,r} + N_{1\,s} + N_{2\,m} - N_{h\,m} - [(e_{1\,s} + \cdots + e_{1\,m-1}) + (e_{3\,r} + \cdots + e_{3\,m-1}) + \cdots + (e_{h\,r} + \cdots + e_{h\,m-1}) + (f_{r\,s} + \cdots + f_{r\,m}) + \cdots + (f_{m-2\,m-1} + f_{m-2\,m}) + f_{m-1\,m}]$; $h < r < s \leqslant m$, *where if* $s = m$ *we mean that* $(e_{1\,s} + \cdots + e_{1\,m-1})$ *does not appear.*

(4) $N_{r\,s} = N_{r\,h} + M_{s\,m} - M_{h\,m} - [(e_{3\,h} + \cdots + e_{3\,s-1}) + \cdots + (e_{r\,h} + \cdots + e_{r\,s-1}) + (f_{h\,m} + \cdots + f_{s-1\,m})]$; $r < h < s < m$.

(5) $M_{r\,s} = M_{r\,m} + N_{h\,s} - N_{h\,m} - [(e_{1\,s} + \cdots + e_{1\,m-1}) + (e_{2\,s} + \cdots + e_{2\,m-1}) + (e_{3\,s} + \cdots + e_{3\,m-1}) + \cdots + (e_{h\,s} + \cdots + e_{h\,m-1}) + (f_{r\,s} + \cdots + f_{r\,m-1}) + \cdots + (f_{s-1\,s} + \cdots + f_{s-1\,m-1}) + (2f_{s\,s+1} + \cdots + 2f_{s\,m-1} + f_{s\,m}) + \cdots + f_{m-1\,m}]$; $r < s < m$, $h < s < m$.

(6) $M_{r\,s} = M_{r\,m} + M_{h\,s} - M_{h\,m} - [(f_{r\,s} + \cdots + f_{r\,m-1}) + \cdots + (f_{h-1\,s} + \cdots + f_{h-1\,m-1})]$; $r < h \leqslant s < m$, *where if* $h = s$ *we substitute* $M_{h\,s}$ *with* $N_{1\,s} + N_{2\,s}$.

(7) $M_{r\,s} = M_{r\,m} + N_{2\,h} + N_{1\,s} - M_{h\,m} - [(e_{1\,s} + \cdots + e_{1\,h-1}) + (f_{r\,s} + \cdots + f_{r\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1})]$; $r < s < h < m$.

(8) $M_{r\,s} = M_{r\,m} + N_{\tau\,s} - N_{\tau\,m} - [(e_{\tau\,s} + \cdots + e_{\tau\,m-1}) + (f_{r\,s} + \cdots + f_{r\,m-1}) + \cdots + f_{m-2\,m-1}]$; $\tau = 1, 2$.

(9) $N_{2\,s} = N_{2\,h} + M_{s\,m} - M_{h\,m} - [(e_{1\,s} + \cdots + e_{1\,h-1}) + (f_{s\,s+1} + \cdots + f_{s\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1})]$; $s < h < m$.

(10) $N_{2\,s} = M_{s\,m} - N_{1\,m} - [(e_{1\,s} + \cdots + e_{1\,m-1}) + (f_{s\,s+1} + \cdots + f_{s\,m-1}) + \cdots + f_{m-2\,m-1}]$; $s < m$.

Note that the identities of Lemmas 6.1 and 6.2 are not independent; we have written them explicitly only for the convenience of the reader to make the proof of 5.3 easier.

Note also that together with each one of the identities where there appears an index 1 or 2, we also have the corresponding identity transformed via the permutation $\sigma = (1, 2)$.

7. Proof of Proposition 5.3, Step 1

We are assuming here that $B < A$, $B' < A'$, and there exists a degeneration $\mathscr{D}'$ such that $B' \leqslant \mathscr{D}'A' < A'$. Moreover we are assuming (of Section 5, sketch of the proof, step 1): (*) No degeneration $\mathscr{D}^*$ on A is trivial with respect to $\mathscr{D}'$; (**) (this second assumption will be given explicitly case by case); and (***) A and B have no common factors.

It follows that we have to consider, separately, the cases when $\mathscr{D}'$ is of type I, II,..., XI (cf. Section 4, Table 3). For each $\mathscr{D}'$ we have to consider all the possible liftings $\mathscr{D}$ which are non-trivial with respect to $\mathscr{D}'$, and realize the program indicated in the sketch of the proof. In the subsections 7.I, 7.II,..., 7.XI, we will consider only the pair $\mathscr{D}', \mathscr{D}$ such that $\mathscr{D}$ is non-trivial (with respect to $\mathscr{D}'$) and we will try to follow the same line of the proof for all the different cases. Unfortunately, as the reader will easily see, the details are not always the same.

7.I. *The Degenerations* $\mathscr{D}'$ *of Type* I

It is easy to see that if $\mathscr{D}'$ is of type I, the only case where we have a non-trivial lifting $\mathscr{D}$ is the one listed in Table I for $(\mathscr{D}', \mathscr{D})$:

Using Table 3, column 3, of Section 4, for the representations of Q_{m-1} and Q_m, we have listed the pairs $(a, b) \in \mathrm{ob}(\mathscr{D}')$ and the pairs $(h, k) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$.

The assumption $B' \leqslant_{\mathscr{D}'} A' < A'$ implies that $N^{A'}_{a\,b} > N^{B'}_{a\,b}$ for every $(a, b) \in \mathrm{ob}(\mathscr{D}')$, and from (5.6) we get:

$$N^A_{a\,b} > N^B_{a\,b}, \qquad \text{for every } (a, b) \in \mathrm{ob}(\mathscr{D}') = H_\alpha \times K_\alpha.$$

The assumption $(**)$ (cf. Section 5, sketch of the proof) is the following:

$$(**) \qquad N^A_{hm} = N^B_{hm} \qquad \text{for some } h \in H_\alpha.$$

We choose an index h such that $(**)$ holds. It follows that

$$(\circ) \qquad N^A_{h\,m-1} - N^A_{h\,m} > N^B_{h\,m-1} - N^B_{h\,m} \geqslant 0.$$

We compute the left-hand side of $(\circ)$ using the identity (1) of Lemma 6.1, and we get the strict inequality:

$$(\circ\circ) \quad (e_{1\,m-1} + e_{2\,m-1} + e_{3\,m-1} + \cdots + e_{h\,m-1} + f_{0\,m-1} + f_{3\,m-1} + \cdots + f_{m-2\,m-1} + f_{m-1\,m})^A > 0.$$

It follows that at least one of the multiplicities of $(\circ\circ)$ for the representation A is strictly positive.

We define now the following sets of indecomposable factors of A:

$$\Delta_0 := \{E_{xy} : e^A_{xy} > 0,\ p \leqslant x \leqslant u-1,\ v+1 \leqslant y \leqslant m-1\}$$

$$\Delta_1 := \{E_{xy} : e^A_{xy} > 0,\ 3 \leqslant x \leqslant p-1,\ v+1 \leqslant y \leqslant m-1\}$$

$$\Delta_2 := \{E_{xy} : f^A_{xy} > 0,\ v+1 \leqslant x \leqslant m-1,\ x < y \leqslant m\}$$

$$\Delta_3 := \{E_{1w} : e^A_{1w} > 0,\ v+1 \leqslant w \leqslant m-1\}$$

$$\Delta_4 := \{E_{2z} : e^A_{2z} > 0,\ v+1 \leqslant z \leqslant m-1\}$$

$$\Delta_5 := \{F_{vy} : f^A_{vy} > 0,\ v < y \leqslant m-1\}$$

$$\Delta_6 := \{F_{xy} : f^A_{xy} > 0,\ 0 \leqslant x \leqslant v-1,\ v < y \leqslant m-1\}.$$

TABLE I for $(\mathscr{L}', \mathscr{L})$

Degeneration	Factors	Type	Obstruction indices
$\mathscr{D}'$ $B' \leqslant_{\mathscr{D}'} A' < A'$	p —— $m-1$ u —— v	I	$(a, b) \in \mathrm{ob}(\mathscr{L}')$ (α) $a \in [p, u-1] = H_\alpha$, $b \in [v+1, m-1] = K_\alpha$
$\mathscr{L}$ $B \nleqslant_{\mathscr{D}} A < A$	p —— m u —— v	I	$(h, \kappa) \in \mathrm{ob}(\mathscr{L}) - \mathrm{ob}(\mathscr{L}')$ (α) $h \in H_\alpha$, $\kappa = m$

Note that $\Lambda_0 = \varnothing$, in fact if $E_{xy} \in \Lambda_0 \neq \varnothing$ we can perform on A the degeneration $\mathscr{D}^*$ of type I on the factors $E_{xy} \oplus E_{uv}$ and $\mathscr{D}^*$ is trivial with respect to $\mathscr{D}'$ (cf. Definition 5.8), against $(*)$.

We set $\Lambda = \bigcup_{\tau=1}^{6} \Lambda_\tau$ and from $(\circ\circ)$ we deduce that $\Lambda \neq \varnothing$.

Next we choose in a suitable way factors in Λ to define a degeneration $\tilde{\mathscr{D}}$ on A and we claim that:

$$(\square) \qquad B \leqslant \tilde{\mathscr{D}} A < A.$$

We collect the various $\tilde{\mathscr{D}}$ which can occur in Table I. In column 1 we define the $\tilde{\mathscr{D}}$'s and draw, for convenience, the factors of A on which $\tilde{\mathscr{D}}$ operates and their type I, II,..., XI (cf. Section 4, Table 3). In column 2 we have computed the pairs $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ and we have specified their type (α), (β), (γ).

Explanation of Table I. We have labelled the $\tilde{\mathscr{D}}$'s by 1, 2, 2′, 2″, 3, 4, and we mean that the label is an order of priority for the choice of $\mathscr{D}$, in the following sense. If $\Lambda_1 \neq \varnothing$ we perform the degeneration 1. If $\Lambda_1 = \varnothing$ and $\Lambda_2 \cup \Lambda_3 \cup \Lambda_4 \neq \varnothing$ we perform 2, 2′, or 2″ according to which of the conditions of column 1 is satisfied. If $\bigcup_{\tau=1}^{4} \Lambda_\tau = \varnothing$ and $\Lambda_5 \neq \varnothing$ we perform 3 and if $\bigcup_{\tau=1}^{5} \Lambda_\tau = \varnothing$ then necessarily $\Lambda_6 \neq \varnothing$ and we perform 4. If $\Lambda_1 \neq \varnothing$ we choose $E_{xy} \in \Lambda_1$, with x maximum. If $\Lambda_1 = \varnothing$ and $\Lambda_2 \cup \Lambda_3 \cup \Lambda_4 \neq \varnothing$, we consider the following factors: $F_{xy} \in \Lambda_2$, y maximum; $E_{1w} \in \Lambda_3$, w maximum; $E_{2z} \in \Lambda_4$, z maximum.

Note that y (resp. w, z) are not defined if Λ_2 (resp. Λ_3, Λ_4) is empty, but at least one of these indices is defined.

If one of the two sets Λ_3, Λ_4 is empty, we may always assume that it is $\Lambda_3 = \varnothing$. If $\Lambda_\tau \neq \varnothing$, $\tau = 3, 4$, we may assume $w \leqslant z$. If $\Lambda_\tau \neq \varnothing$, $\tau = 2, 3, 4$ and $y \geqslant w$, or $\Lambda_2 \neq \varnothing$ and $\Lambda_3 = \varnothing$, we perform the degeneration 2. If $\Lambda_\tau \neq \varnothing$, $\tau = 2, 3, 4$ and $y < w$, or $\Lambda_2 = \varnothing$ and $\Lambda_3 \neq \varnothing$, $\Lambda_4 \neq \varnothing$, we perform the degeneration 2′. If $\Lambda_2 = \Lambda_2 = \varnothing$ and $\Lambda_4 \neq \varnothing$ we perform the degeneration 2″. For the degeneration 3 we choose any $F_{vy} \in \Lambda_5$. For the degeneration 4 we choose $F_{xy} \in \Lambda_6$, with x maximum.

Proof of $(\square)$. We have to verify (5.5)(α), (β), (γ) (cf. Remark 5.4) for all the pairs $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$, which are listed in column 2 of Table I.

If $(r, s) \in \mathrm{ob}(\mathscr{D}) - \mathrm{on}(\mathscr{D}')$ is of type (α), and $r \neq 2$, we consider the identity (1) of Lemma 6.2. We claim that its contribution in brackets is zero where evaluated at the representation A, i.e.,

$$(\circ\circ\circ) \quad [(e_{r+1\,s} + \cdots + e_{r+1\,m-1}) + \cdots + (e_{hs} + \cdots + e_{h\,m-2})]^A = [\cdots]^A = 0$$

for all the $\tilde{\mathscr{D}}$'s.

In fact $(\circ\circ\circ)$ holds in 1: as $\Lambda_0 = \varnothing$ and x is maximum; in 2, 2′, 2″, 4: as $\Lambda_0 \cup \Lambda_1 = \varnothing$.

TABLE I for $\tilde{\mathscr{D}}$

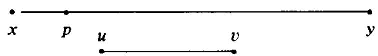
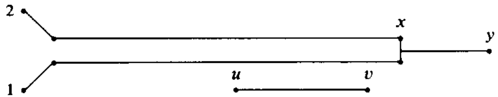
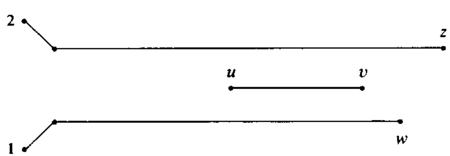
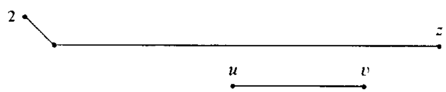
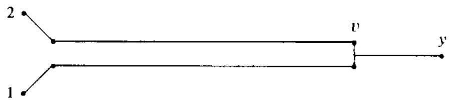
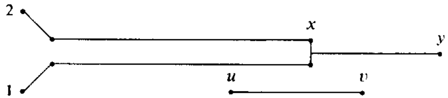
Column 1				Column 2		
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$\Lambda_1 \neq \varnothing$	$E_{xy} \in \Lambda_1$ x max	1		I	(α) $r \in [x, p-1]$,	$s \in [v+1, y]$
$\Lambda_1 = \varnothing$ $\bigcup_{\tau=2}^{4} \Lambda_\tau \neq \varnothing$	$F_{xy} \in \Lambda_2$ y max $E_{2w} \in \Lambda_3$ w max	2	 $y \geqslant \min(w, z) = w$, or $\Lambda_3 = \varnothing$	V	(α) $r \in [0, p-1]$, (β) $r \in [v+1, x]$,	$s \in [v+1, x]$ $s \in [y+1, m]$
	$E_{2z} \in \Lambda_4$ z max	2′	 $y < \min(w, z) = w$, or $\Lambda_2 = \varnothing$	XI	(α) $\begin{cases} r = 2, \\ r = [0, p-1], \end{cases}$ (β) $r \in [v+1, z]$,	$s \in [w+1, z]$ $s \in [v+1, z]$ $s \in [w+1, m]$

Table continued

TABLE I for $\tilde{\mathscr{D}}$ (*continued*)

Column 1				Column 2			
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type		$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{L}')$	
		2″		II	(α)	$\begin{cases} r = 2, \\ r \in [0, p-1], \end{cases}$	$s \in [v+1, z]$ $s \in [v+1, z]$
					(β)	$r \in [v+1, z]$,	$s \in [v+2, m]$
			$\Lambda_2 = \Lambda_3 = \varnothing$				
$\bigcup_{\tau=1}^{4} \Lambda_\tau = \varnothing$ $\Lambda_5 \neq \varnothing$	$F_{vy} \in \Lambda_5$	3		IX	(α)	$r = 2$,	$s \in [v+1, y]$
					(β)	$r = [v+1, y-1]$,	$s \in [v+2, y]$
$\bigcup_{\tau=1}^{5} \Lambda_\tau = \varnothing$ $\Lambda_6 \neq \varnothing$	$F_{xy} \in \Lambda_6$ x max	4		III	(α)	$r \in [0, p-1] \cup \{1, 2\}$,	$s \in [v+1, y]$
					(β)	$\begin{cases} r \in [x+1, v], \\ r \in [v+1, y], \end{cases}$	$s \in [v+1, y]$ $s \in [y+1, m]$
					(γ)	$r \in [v+1, y-1]$,	$s \in [v+2, y]$

We use the same identity (1) of Lemma 6.2 for both the representations A and B and we get

$$N^A_{rs} = \{N_{rm} + N_{hs} - N_{hm}[\cdots]\}^A > \{N_{rm} + N_{hs} - N_{hm} - [\cdots]\}^B = N^B_{rs}$$

as a consequence of $(**)$, $(\circ\circ\circ)$, and the inequalities:

$$N^A_{rm} \geqslant N^B_{rm} \qquad (B < A); \qquad N^A_{hs} > N^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

If $(2, s) \in \mathrm{ob}(\mathscr{D}) - \mathrm{on}(\mathscr{D}')$ (of type (α)), we consider the identity (2) of Lemma 6.2. Its contribution in brackets is zero when evaluated ay A, in fact it is the sum of the following contributions: $(e_{1s} + \cdots e_{1\ m-1})^A = 0$ for $2'$ as either $A_3 = \varnothing$ or $s > w$ and w is maximum; for $2''$, 3, 4 as $A_3 = \varnothing$; $((e_{3r} + \cdots + e_{3\ m+1}) + \cdots + (e_{hr} + \cdots + e_{h\ m-1}))^A = 0$ for $2'$, 3, 4 as $A_0 \cup A_1 = \varnothing$; $((f_{h\ s+1} + \cdots + f_{h\ m}) + \cdots + f_{m-1\ m})^A = 0$ for $2'$, $2''$ as $A_2 = \varnothing$ or $s > w > y$, and y is maximum; for 3, 4 as $A_2 = \varnothing$. If $(1, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ (in this case $\tilde{\mathscr{D}}$ is 4) we reproduce the same argument using the permutation $\sigma = (1, 2)$. It follows:

$$(\circ\circ\circ) \qquad [\cdots]^A = 0.$$

We use the same identity for both the representations A and B and we get:

$$N^A_{2s} = \{N_{2m} + N_{hs} + N_{hm} - [\cdots]\}^A > \{N_{2m} + N_{hs} + N_{hm} - [\cdots]\}^B = N^B_{2s},$$

as a consequence of $(**)$, $(\circ\circ\circ)$, and the inequalities:

$$N^A_{2m} \geqslant N^B_{2m} \qquad (B < A); \qquad N^A_{hs} > N^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

Similarly $N^A_{1s} > N^B_{1s}$ and (5.5)(α) is proved.

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $(r, s) \notin [x+1, v] \times [v+1, y]$, we use the identity (3) of Lemma (6.2). Its contribution $[\cdots]$ is zero when evaluated at A. In fact $[\cdots]^A$ is the sum of the following contributions: $(e_{1s} + \cdots + e_{1\ m-1})^A = 0$ in 2 as $y > w$ and w is max; in $2'$ as w is max; in $2''$, 3, 4 as $A_3 = \varnothing$; $((e_{2r} + \cdots + e_{3\ m-1}) + \cdots + (e_{hr} + \cdots + e_{h\ m-1}))^A = 0$ as $A_0 \cup A_1 = \varnothing$; $((f_{rs} + \cdots + f_{rm}) + \cdots + f_{m-1\ m})^A = 0$ in 2 as y is max; in $2'$ as $w > y$, y max; in $2''$, 3, 4 as $A_2 = \varnothing$. If follows that for the identity (3) of Lemma 6.2

$$(\circ\circ\circ) \qquad [\cdots]^A = 0.$$

We use the same identity for the representations A, B. We get:

$$\begin{aligned} M^A_{rs} &= \{N_{hr} - N_{hm} + N_{1s} + N_{2m} - [\cdots]\}^A \\ &> \{N_{hr} - N_{hm} + N_{1s} + N_{2m} - [\cdots]\}^B = M^B_{rs} \end{aligned}$$

as a consequence of $(**)$, $(\circ\circ\circ)$, and the inequalities:

$$N_{1s}^A \geqslant N_{1s}^B, \qquad N_{2m}^A \geqslant N_{2m}^B \qquad (B < A); \qquad N_{hr}^A > N_{hr}^B \qquad ((h, r) \in \mathrm{ob}(\mathscr{D}')).$$

If (r, s) is of type (β) and $r \in [x+1, v]$, $s \in [v+1, y]$ (degeneration 4) we consider the identity (5) of Lemma 6.2. Its brackets is zero where evaluated at A, i.e.,

$$(\circ\circ\circ) \qquad\qquad [\cdots]^A$$

holds for the identity (5), as a consequence of the fact that this type of index only occurs for the degeneration 4, and $\bigcup_{\tau=0}^{5} \Lambda_\tau = \varnothing$. We use the same identity for the representations A, B and we get $M_{rs}^A > M_{rs}^B$ as a consequence of $(**)$, $(\circ\circ\circ)$, and the inequalities:

$$M_{rm}^A \geqslant M_{rm}^B \qquad (B < A); \qquad N_{hs}^A > N_{hs}^B \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

Equation (5.5)(β) is now proved.

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (γ) (degeneration 4), we use the identity (3) of Lemma 6.2, and we have for its brackets:

$$(\circ\circ\circ) \qquad\qquad [\cdots]^A = 0$$

as a consequence of the fact that this type of index only occurs for the degeneration 4 and $\bigcup_{\tau=0}^{5} \Lambda_\tau = \varnothing$.

We use the same identity for the representations A, B and we get $M_{rs}^A > M_{rs}^B + 1$, as a consequence of $(**)$, $(\circ\circ\circ)$, and the inequalities:

$$N_{1s}^A > N_{1s}^B \qquad ((1, s) \in \underset{\text{type } \alpha}{\mathrm{ob}} (\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')); \qquad N_{2m}^A \geqslant N_{2m}^B$$
$$(B < A); \qquad N_{hs}^A > N_{hs}^B \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

Step 1 of Proposition 5.3 is now proved for the degenerations $\mathscr{D}'$ of type I.

Before proceeding with the other possible types for $\mathscr{D}'$, we want to point out the ingredients we used for $\mathscr{D}'$ of type I, and what is common or varies for all the other cases.

The assumptions $(*)$ and $(***)$ are common to all the types.

The assumption $(**)$ does depend on the pair $(\mathscr{D}', \mathscr{D})$ and we will specify it case by case, therefore we will have several ineqùalities $(\circ)$, but in any case we will deduce a strict inequality $(\circ\circ)$ using a suitable identity of Lemma 6.1.

Next we will define a suitable set Λ of factors of A and we will use $(\circ\circ)$ to claim that $\Lambda \neq \varnothing$.

We will use the set Λ to define some degenerations $\tilde{\mathscr{D}}$, collected in tables, and the claim will always be

$$(\square) \qquad B \leqslant \tilde{\mathscr{D}}A < A.$$

We will explain the tables, if necessary, and we will prove ($\square$), i.e., (5.5)(α), (β), (γ), for all the pairs $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$.

To do this we will always consider a suitable identity of Lemma 6.2 and consider its bracket contribution (cf. Lemma 6.2) evaluated at the representation A. We will prove that

$$(\circ\circ\circ) \qquad [\cdots[^A = 0.$$

Then we will consider the same identity for both the representations A and B to deduce the inequalities of (5.5).

From now on we will try to be as short as possible, but in such a way that our argument still is understandable.

7.II. *The Degenerations* $\mathscr{D}'$ *of Type* II

We first consider the pairs $(\mathscr{D}', \mathscr{D})$; see Table II_1 for $(\mathscr{D}', \mathscr{D})$. The assumption $(**)$ is:

$$(**) \qquad M_{hm}^A = M_{hm}^B \qquad \text{for some } h \in H_\beta.$$

We choose an index h such that $(**)$ holds, and we have:

$$(\circ) \qquad [N_{\tau m} + N_{0h} - M_{hm}]^A > [N_{\tau m} + N_{0h} - M_{hm}]^B, \qquad \tau = 1, 2$$

as a consequence of ($\circ$) and of the inequalities

$$N_{\tau m}^A \geqslant N_{\tau m}^B, \qquad \tau = 1, 2 \text{ as } A > B$$

$$N_{0h}^A > N_{0h}^B, \qquad \text{as } 0 \in H_\alpha,\ h \in K_\alpha.$$

From the identity (2) of Lemma 6.1 and ($\circ$) we deduce

$$(\circ\circ) \qquad [e_{\tau m} + f_{hm} + f_{h+1\, m} + \cdots + f_{m-1\, m}]^A > 0, \qquad \tau = 1, 2.$$

We define the following sets of indecomposable factors of A:

$\Lambda_1 = \{E_{xy} : e_{xy}^A > 0, 3 \leqslant x \leqslant u-1, v+1 \leqslant y \leqslant m-1\}$

$\Lambda_2 = \{F_{wm} : f_{wm}^A > 0, h \leqslant w \leqslant m-1\}$

$\Lambda_3 = \{E_{1m} \oplus E_{2m} : e_{1m}^A > 0, e_{2m}^A > 0\}$

and we set $\Lambda = \bigcup_{\tau=1}^3 \Lambda_\tau$. From ($\circ\circ$) it follows that $\Lambda \neq \varnothing$. (See Table II_1 for $\tilde{\mathscr{D}}$.)

TABLE II_1 for $(\mathscr{D}', \mathscr{D})$

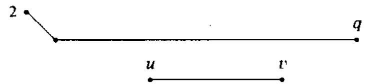
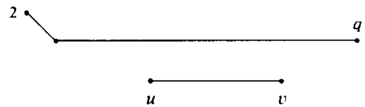
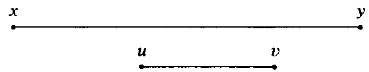
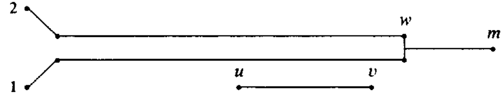
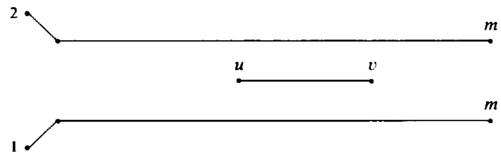
Degeneration	Factors	Type	Obstruction indices
$\mathscr{D}'$ $B' \leqslant_{\mathscr{D}'} A' < A'$	 $q \leqslant m-1$	II	$(a, b) \in \mathrm{ob}(\mathscr{D}')$ (α) $a \in [0, u-1] \cup \{2\} = H_\alpha$, $b \in [v+1, q] = K_\alpha$ (β) $a \in [v+1, q] = H_\beta$, $b \in [v+2, m-1] = K_\beta$
$\mathscr{D}$ $B \nleqslant_{\mathscr{D}} A < A$		II	$(h, k) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$ (β) $h \in H_\beta$, $\kappa = m$

TABLE II_1 for $\tilde{\mathscr{D}}$

			Column 1	Column 2	
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$
$\Lambda_1 \cup \Lambda_2 \neq \varnothing$	$E_{xy} \in \Lambda_1$ y min $E_{wm} \in \Lambda_2$ w min	1	 $q < y \leqslant w, q < m-1$	I	(α) $r \in [x, u-1]$, $s \in [q+1, y]$
		1′	 $q < w < y, q < m-1$	V	(α) $r \in [0, u-1]$, $s \in [q+1, w]$
$\Lambda_1 \cup \Lambda_2 = \varnothing$ $\Lambda_3 \neq \varnothing$		2	 $q \leqslant m-1$	XI	(α) $r \in [0, u-1]$, $s \in [q+1, m]$

Explanation of Table II_1. If Λ_2 (or Λ_1) is empty but $\Lambda_1 \cup \Lambda_2 \neq \emptyset$, then w is not defined (resp. y), we mean that we perform the degeneration $\tilde{\mathscr{D}}$ labelled by 1 (resp. 1′).

If $\Lambda_1 \neq \emptyset$ and $\Lambda_2 \neq \emptyset$ and $y \leqslant w$ we must also have $q < y \leqslant w$, otherwise $\tilde{\mathscr{D}}$ is trivial with respect to $\mathscr{D}'$, and this is the case we perform 1; similarly if $y > w$ we must also have $w > q$ and we perform 1′.

Proof of (□). We only have to verify (5.5)(α). We consider the identity (4) of Lemma 6.2, and its contribution in brackets is zero when evaluated at A, i.e., (∘∘∘) holds.

In fact $[\cdots]^A$ is the sum of the following contributions: $((e_{3h} + \cdots + e_{3\,s-1}) + \cdots + (e_{rh} + \cdots + e_{r\,s-1}))^A = 0$ for 1 as y is minimum; for 1′ as y is minimum and $w < y$ or $\Lambda_1 = \emptyset$; for 2 as $\Lambda_1 = \emptyset$. $(f_{hm} + \cdots + f_{s-1\,m})^A = 0$ for 1 as w is minimum and $y \leqslant w$ (or $\Lambda_2 = \emptyset$); for 1′ as w is minimum; for 2 as $\Lambda_2 = \emptyset$. From the identity (4) of Lemma 6.2 applied to A and B we deduce (5.5)(α), as in this case we have the assumption (∗∗), (∘∘∘), and

$$M^A_{sm} \geqslant M^B_{sm} \qquad (B < A); \quad N^A_{rh} > N^B_{rh} \qquad ((r,h) \in \mathrm{ob}(\mathscr{D}')).$$

Note that if $q = m-1$, the previous discussion is simplified. In fact is this case the degenerations $\tilde{\mathscr{D}}1$ and 1′ are trivial and we only have the possibility 2.

The other pairs $(\mathscr{D}', \mathscr{D})$, $\mathscr{D}'$ of type II we still have to consider are defined in Table II_2 for $(\mathscr{D}', \mathscr{D})$. Our assumption (∗∗) is now that one of the following equalities holds for at least an index h:

$$(**) \qquad \begin{array}{lll} \text{(i)} & M^A_{hm} = M^B_{hm}, & h \in H_\beta \\ \text{(ii)} & N^A_{hm} = N^B_{hm}, & h \in H_\alpha \;\; h \neq 2 \\ \text{(iii)} & N^A_{2m} = N^B_{2m}, & h = 2 \in H_\alpha. \end{array}$$

If the assumption (∗∗) is

$$\text{(i)} \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta$$

we reproduce the same argument as for $\mathscr{D}$ acting on the factors $E_{2\,m-1} \oplus E_{uv}$, and we use the same set Λ and Table II_1. Therefore Proposition 5.3 is proved under the assumption (∗∗)(i).

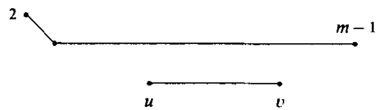
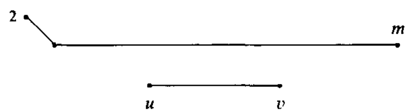
It follows that we may assume

$$M^A_{h'm} > M^B_{h'm} \qquad \text{for every } h' \in H_\beta$$

and (∗∗) is

$$\text{(ii)} \qquad N^A_{hm} = N^B_{hm} \qquad \text{for some } h \in H_\alpha,\ h \neq 2.$$

TABLE II_2 for $(\mathscr{D}', \mathscr{D})$

Degeneration	Factors	Type		Obstruction indices	
$\mathscr{D}'$				$(a, b) \in \mathrm{ob}(\mathscr{D}')$	
$B' \leqslant_{\mathscr{D}'} A' < A'$		II	(α)	$a \in [0, u-1] \cup \{2\} = H_\alpha$,	$b \in [v+1, m-1] = K_\alpha$
			(β)	$a \in [v+1, m-1] = H_\beta$,	$b \in [v+2, m-1] = K_\beta$
$\mathscr{D}$				$(h, \kappa) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$	
$B \nleqslant_{\mathscr{D}} A < A$		II	(α)	$h \in H_\alpha$,	$\kappa = m$
			(β)	$h \in H_\beta$,	$\kappa = m$

We choose an index h such that $(**)$(ii) holds and we have

$$(\circ) \qquad N^A_{h\ m-1} - N^A_{hm} > N^B_{h\ m-1} - N^B_{hm} \geqslant 0$$

as a consequence of $(**)$ and the inequality $N^A_{h\ m-1} > N^B_{h\ m-1}$ as $h \in H_\alpha$, $m-1 \in K_\alpha$.

From the identity (1) of Lemma 6.1, applied to the representation A and from $(\circ)$ we deduce:

$$(\circ\circ) \qquad [(e_{1\ m-1} + e_{2\ m-1} + e_{3\ m-1} + \cdots + e_{h\ m-1}) \\ + (f_{0\ m-1} + f_{3\ m-1} + \cdots + f_{m-2\ m-1} + f_{m-1\ m})]^A > 0.$$

We define the following sets of indecomposable factors of A.

$$\Lambda_0 = \{E_{xy} : e^A_{xy} > 0, 3 \leqslant x \leqslant h, v+1 \leqslant y \leqslant m-1\}$$
$$\Lambda_1 = \{F_{vy} : f^A_{vy} > 0, v+1 \leqslant y \leqslant m-1\}$$
$$\Lambda_2 = \{E_{2y} : e^A_{2y} > 0, v+1 \leqslant y \leqslant m-1\}$$
$$\Lambda_3 = \{F_{xy} : f^A_{xy} > 0, v+1 \leqslant x \leqslant m-1, x < y \leqslant m\}$$
$$\Lambda_4 = \{E_{1y} : e^A_{1y} > 0, v+1 \leqslant y \leqslant m-1\}$$
$$\Lambda_5 = \{F_{xy} : f^A_{xy} > 0, 0 \leqslant x \leqslant v-1, v+1 \leqslant y \leqslant m-1\}.$$

Note that $\Lambda_0 = \varnothing$, otherwise we can perform on A the degeneration $\mathscr{D}^*$ of type I to the factors $E_{xy} \oplus E_{uv}$, which is trivial with respect to $\mathscr{D}'$, against $(*)$.

Similarly $\Lambda_1 = \varnothing$, otherwise we can perform the degeneration $\mathscr{D}^*$: $F_{vy} \mapsto E_{2v} \oplus E_{1y}$ (type IX), which is trivial.

We set $\Lambda = \bigcup_{\tau=2}^{5} \Lambda_\tau$ and from $(\circ\circ)$ it follows that $\Lambda \neq \varnothing$. (See Table II$_2$ for $\tilde{\mathscr{D}}$ assumption $(**)$(ii).)

Proof of ($\square$). For the pair of indices (r, s) of type (β) with $s = m$ ($\square$) trivially holds as we are assuming

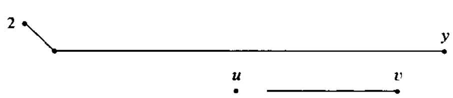
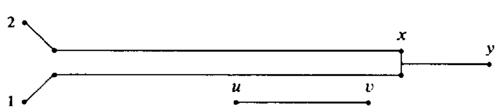
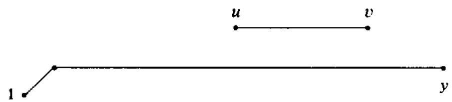
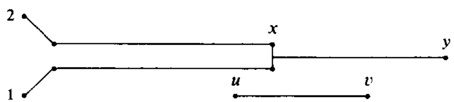
$$M^A_{h'm} > M^B_{h'm} \qquad \text{for every } h' \in H_\beta$$

and $r \in H_\beta$.

Therefore we have to prove ($\square$) only for the remaining pairs of the degeneration 3 and 4.

For the pairs (r, s) of type (α) we consider the identity (2) of Lemma 6.2, transformed via the permutation $\sigma = (1, 2)$. Its contribution in brackets is

TABLE II_2 for $\tilde{\mathscr{D}}$: Assumption (**)(ii)

	Column 1			Column 2		
	Choice in Λ		$\tilde{\mathscr{D}} : B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$\Lambda_2 \neq \varnothing$	$E_{2y} \in \Lambda_2$	1		II (β)	$r \in [v+1, y]$,	$s = m$
$\Lambda_2 = \varnothing$ $\Lambda_3 \neq \varnothing$	$F_{xy} \in \Lambda_3$	2		V (β)	$r \in [v+1, y]$,	$s = m$
$\bigcup_{\tau=2}^{3} \Lambda_\tau = \varnothing$ $\Lambda_4 \neq \varnothing$	$E_{1y} \in \Lambda_4$	3		II (α) (β)	$r = 1$, $r \in [v+1, y]$,	$s \in [v+1, y]$ $s = m$
$\bigcup_{r=2}^{4} \Lambda_\tau = \varnothing$ $\Lambda_5 \neq \varnothing$	$F_{xy} \in \Lambda_5$ x max	4		III (α) (β) (γ)	$r = 1$, $\begin{cases} r \in [x+1, v], \\ r \in [v+1, y], \end{cases}$ $r \in [v+1, y-1]$,	$s \in [v+1, y]$ $s \in [v+1, y]$ $s = m$ $s \in [v+2, y]$

zero when evaluated at the representation A, i.e., (∘∘∘) holds. In fact we have:

$((e_{2s} + \cdots + e_{2\,m-1}))^A = 0$ as $\Lambda_2 \neq \varnothing$;

$((e_{3s} + \cdots e_{3\,m-1}) + \cdots + (e_{hs} + \cdots + e_{h\,m-1}))^A = 0$ as $\Lambda_0 = \varnothing$;

$((f_{s\,s+1} + \cdots + f_{s\,m-1}) + \cdots + f_{m-2\,m-1} + f_{m-1\,m})^A = 0$ as $\Lambda_3 = \varnothing$.

Therefore we have

$$(\circ\circ\circ) \qquad [\cdots]^A = 0.$$

Using the same identity for both the representations A and B we get $N^A_{1s} > N^B_{1s}$ as a consequence of $(**)$(ii) and of the inequalities:

$$N^A_{1m} \geqslant N^N_{1m} \qquad (B < A); \qquad N^A_{hs} > N^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

For the pairs (r, s) of type (β), with $r \in [x+1, v]$, $s \in [v+1, y]$, we use the identity (5) of Lemma 6.2. Its contribution in brackets is zero when evaluated at A, i.e., (∘∘∘) holds, as it is the sum of the following terms:

$(e_{1s} + \cdots + e_{1\,m-1})^A = 0$ as $\Lambda_4 = \varnothing$;

$(e_{2s} + \cdots + e_{2\,m-1})^A = 0$ as $\Lambda_2 = \varnothing$;

$((e_{3s} + \cdots + e_{3\,m-1}) + \cdots + (e_{rs} + \cdots + e_{r\,m-1}))^A = 0$ as $\Lambda_0 = \varnothing$;

$((f_{r\,r+1} + \cdots + f_{r\,m-1}) + \cdots + (f_{v-1\,v} + \cdots + f_{v-1\,m-1}))^A = 0$ as x is maximum;

$(f_{v\,v+1} + \cdots + f_{v\,m-1})^A = 0$ as $\Lambda_1 = \varnothing$;

$(\cdots + (2f_{s\,s+1} + \cdots + f_{s\,m}) + \cdots + f_{m-1\,m})^A = 0$ as $\Lambda_3 = \varnothing$.

We use the identity (5) for both the representations A and B and we get $M^A_{rs} > M^B_{rs}$, as a consequence of (∘∘∘), $(**)$(ii), and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad N^A_{hs} > N^B_{hs} \qquad ((h, s) \in H_\alpha \times K_\alpha \subset \mathrm{ob}(\mathscr{D}')).$$

For the pairs (r, s) of type (γ) we use the same identity (5). Again (∘∘∘) holds, this time only because $\bigcup_{\tau=1}^4 \Lambda_\tau = \varnothing$. Using the identity (5) both for A and B we get

$$M^A_{rs} > M^B_{rs} + 1$$

as a consequence of (∘∘∘), $(**)$(ii), and the following inequalities:

$$M^A_{rm} > M^B_{rm} \qquad ((*)\text{(i) has been proved});$$

$$N^A_{hs} > N^B_{hs} \qquad ((h, s) \in H_\alpha \times K_\alpha \subset \mathrm{ob}(\mathscr{D}')).$$

TABLE II_2 for $\tilde{\mathscr{D}}$: Assumption (**)(iii)

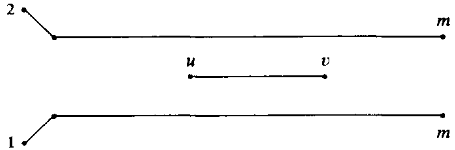
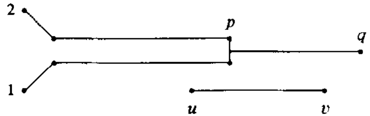
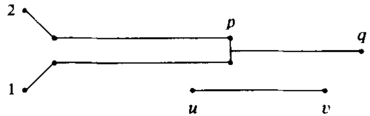
	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$
$\tilde{\mathscr{D}}$: 	XI	(α) $r \in [0, u-1]$, $s = m$

TABLE III_1 for $(\mathscr{D}', \mathscr{D})$

Degeneration	Factors	Type	Obstruction indices
$\mathscr{D}'$ $B' \leqslant \mathscr{D}'A' < A'$	 $q \leqslant m-1$	III	$(a, b) \in \mathrm{ob}(\mathscr{D}')$ (α) $a \in [0, u-1] \cup \{1, 2\} = H_\alpha$, $b \in [v+1, q] = K_\alpha$ (β) $\begin{cases} a \in [p+1, v] = H'_\beta, & b \in [v+1, q] = K'_\beta \\ a \in [v+1, q] = H''_\beta, & b \in [q+1, m-1] = K''_\beta \end{cases}$ (γ) $a \in [v+1, q-1] = H_\gamma$, $b \in [v+2, q] = K$
$\mathscr{D}$ $B \nleqslant \mathscr{D}A < A$		III	$(h, \kappa) \in \mathrm{ob}(\mathscr{L}) - \mathrm{ob}(\mathscr{L}')$ (β) $h \in H''_\beta$, $\kappa = m$

Proposition 5.3 is now proved under the assumption $(**)$(i) and (ii). Therefore we may assume

$$M^A_{h'm} > M^B_{h'm} \qquad (h' \in H_\beta); \qquad N^A_{h'm} > N^B_{h'm} \qquad (h' \in H_\alpha, h' \neq 2)$$

and we must assume $(**)$ of the form:

(iii) $$N^A_{2m} = N^B_{2m}.$$

We have

($\circ$) $$N^A_{0m} - N^A_{2m} > N^B_{0m} - N^B_{2m}$$

as a consequence of $(**)$(iii) and the fact that $0 \in H_\alpha$. From the identity (4) of Lemma 6.1 (transformed with the permutation $\sigma = (1, 2)$) and from ($\circ$)we deduce:

($\circ\circ$) $$e^A_{1m} = 0$$

and $\tilde{\mathscr{Q}}$ is defined in Table II_2 assumption $(**)$(iii).

($\square$) is trivially proved, as $r \in H_\alpha$, $r \neq 2$.

7.III. *The Degenerations* $\mathscr{Q}'$ *of Type* III

We first consider the pairs $(\mathscr{Q}', \mathscr{Q})$ defined in Table III_1. The assumption $(**)$ is:

$(**)$ $$M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H''_\beta.$$

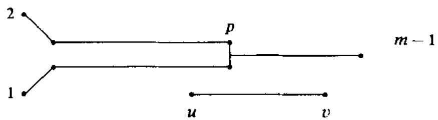
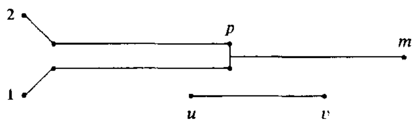
We claim that we can reproduce the same argument of 7.II ($q \leqslant m-1$). In fact the obstruction indices of $\mathscr{Q}'$ in 7.II are a subset of the obstruction indices here; moreover the degenerations $\tilde{\mathscr{Q}}$ of Table II_1 never use the factor E_{2q}.

The remaining pairs $\mathscr{Q}'$, $\mathscr{Q}$ we have to consider are defined in Table III_2 for $(\mathscr{Q}', \mathscr{Q})$. The assumption $(**)$ is now that one of the following equalities holds for at least an index h:

$$(**) \qquad \begin{array}{lll} \text{(i)} & M^A_{hm} = M^B_{hm}, & h \in H_\gamma \\ \text{(ii)} & M^A_{hm} = M^B_{hm} + 1, & h \in H_\gamma \\ \text{(iii)} & M^A_{hm} = M^B_{hm}, & h \in H_\beta \\ \text{(iv)} & N^A_{hm} = N^B_{hm}, & h \in H_\alpha,\ h \neq 1, 2 \\ \text{(v)} & N^A_{\tau m} = N^B_{\tau m} & \tau = 1, 2. \end{array}$$

If the assumption $(**)$ is (i) for some $h \in H_\gamma$, we reproduce the same argument as for the pairs $(\mathscr{Q}', \mathscr{Q})$ with $q \leqslant m-1$, and Proposition 5.3 is proved under such assumption.

TABLE III_2 for $(\mathscr{D}', \mathscr{D})$

Degeneration	Factors	Type	Obstruction indices	
$\mathscr{D}'$		III	$(a, b) \in \mathrm{ob}(\mathscr{D}')$	
$B' \leqslant_{\mathscr{D}'} A' < A'$			(α) $a \in [0, u-1] \cup \{1, 2\} = H_\alpha$,	$b \in [v+1, m-1] = K_\alpha$
			(β) $a \in [p+1, v] = H_\beta$,	$b \in [v+1, m-1] = K_\beta$
			(γ) $a \in [v+1, m-1] = H_\gamma$,	$b \in [v+2, m-1] = K$
$\mathscr{D}$		III	$(h, \kappa) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$	
$B \not\leqslant_{\mathscr{D}} A < A$			(α) $h \in H_\alpha$,	$\kappa = m$
			(β) $h \in H_\beta$,	$\kappa = m$
			(γ) $h \in H_\gamma$,	$\kappa = m$

Therefore we may assume $M^A_{h'm} > M^B_{h'm}$ for every $h' \in H_\gamma$ and (**)(ii):

$$M^A_{hm} = M^B_{hm} + 1 \qquad \text{for some } h \in H_\gamma.$$

We choose an index h such that (**)(ii) holds. We have:

$$(\circ) \qquad M^A_{h\ m-1} - M^A_{hm} > M^B_{h\ m-1} - M^B_{hm}.$$

In fact if $h < m-1$, $(h, m-1) \in H_\gamma \times K_\gamma \subset \mathrm{ob}(\mathscr{D}')$, i.e., $M^A_{h\,m-1} > M^B_{h\ m-1} + 1$; if $h = m-1$ we have $N^A_{\tau\ m-1} > N^B_{\tau\ m-1}$, $\tau = 1, 2$, as $(\tau, m-1) \in H_\alpha \times K_\alpha$. From the identity (5) of Lemma 6.1 we deduce:

$$(\circ\circ) \qquad [f_{0\ m-1} + f_{3\ m-1} + \cdots + f_{h-1\ m-1}]^A > 0.$$

We define the following sets of indecomposable factors of A

$$\Lambda_0 = \{F^A_{vy} : f^A_{vy} > 0, v+1 \leqslant y \leqslant m-1\}$$

$$\Lambda_1 = \{E_{1w} : e^A_{1w} > 0, v+1 \leqslant w \leqslant m-1\}$$

$$\Lambda_2 = \{E_{2w} : e^A_{2w} > 0, v+1 \leqslant w \leqslant m-1\}$$

$$\Lambda_3 = \{F_{xy} : f^A_{xy} > 0, p \leqslant x \leqslant v-1, v+1 \leqslant y \leqslant m-1\}$$

$$\Lambda_4 = \{F_{xy} : f^A_{xy} > 0, v+1 \leqslant x \leqslant m-1, x < y \leqslant m\}$$

$$\Lambda_5 = \{F_{xy} : f^A_{xy} > 0, 0 \leqslant x \leqslant p-1, v+1 \leqslant y \leqslant m-1\}.$$

Note that $\Lambda_0 = \varnothing$, otherwise we could perform a degeneration $\mathscr{D}^*$ of type IX on $f_{vy} \in \Lambda_0$, against (*).

We set $\Lambda = \bigcup_{\tau=1}^{5} \Lambda_\tau$ and from $(\circ\circ)$ we have $\Lambda \neq \varnothing$. (See Table III$_2$ for $\tilde{\mathscr{D}}$ assumption (**)(ii).)

Proof of ($\square$). For $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ and $s = m$, ($\square$) holds as a consequence of the fact that $r \in H_\gamma$ and $M^A_{hm} > M^B_{hm}$ for every $h \in H_\gamma$. For the remaining pairs (r, s), i.e., $r \in [x+1, p]$, $s \in [v+1, y]$ (degeneration 4), we may have $s \geqslant h$ or $s < h$. If $s \geqslant h$, we consider the identity (6) of Lemma 6.2. $(\circ\circ\circ)$ holds as x is maximum and $\Lambda_0 \cup \Lambda_3 \cap \Lambda_4 = \varnothing$, and (5.5)($\beta$) follows (**)(ii), $(\circ\circ\circ)$, and the inequalities:

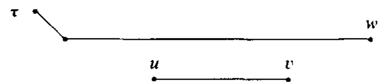
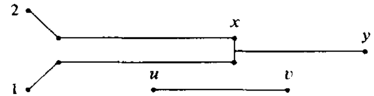
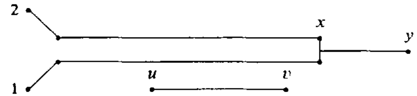
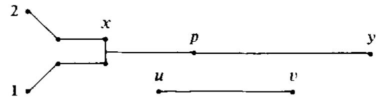
$$M^A_{rm} \geqslant M^B_{rm} \quad (B < A); \qquad M^A_{hs} > M^B_{hs} + 1 \quad ((h, s) \in H_\gamma \times K_\gamma \subset \mathrm{ob}(\mathscr{D}'));$$

$$(\text{or } N^A_{\tau,s} > N^B_{\tau,s}, \qquad \tau = 1, 2, s = h).$$

Is $s < h$ we consider the identity (7) of Lemma 6.2. $(\circ\circ\circ)$ holds as $\Lambda_1 = \varnothing$, x is maximum, and $\Lambda_0 \cup \Lambda_3 \cup \Lambda_4 = \varnothing$, and (5.5)($\beta$) follows now from (**)(ii), $(\circ\circ\circ)$, and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \quad (B < A); \qquad N^A_{2h} > N^B_{2h} \quad (h \in K_\alpha); \qquad N^A_{1s} > N^B_{1s} \quad (s \in K_\alpha).$$

TABLE III_2 for $\tilde{\mathscr{D}}$: Assumption $(**)$(ii)

Column 1				Column 2		
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$\Lambda_\tau \neq \varnothing$, $\tau = 1,2$	$E_{\tau w} \in \Lambda_\tau$	1		II	(β) $r \in [v+1, w]$,	$s = m$
$\bigcup_{\tau=1}^{2} \Lambda_\tau = \varnothing$, $\Lambda_3 \neq \varnothing$	$F_{xy} \in \Lambda_3$	2		III	(β) $r \in [v+1, y]$,	$s = m$
$\bigcup_{\tau=1}^{3} \Lambda_\tau = \varnothing$, $\Lambda_4 \neq \varnothing$	$F_{xy} \in \Lambda_4$	3		V	(β) $r \in [v+1, x]$,	$s = m$
$\bigcup_{\tau=1}^{4} \Lambda_\tau = \varnothing$, $\Lambda_5 \neq \varnothing$	$F_{xy} \in \Lambda_5$	4		III	(β) $\begin{cases} r \in [x+1, p], \\ r \in [v+1, y], \end{cases}$	$s \in [v+1, y]$ $s = m$

Proposition 5.3 is now proved under the assumption $(**)$(ii). Therefore we may assume that $M^A_{h'm} > M^B_{h'm} + 1$ for every $h' \in H_\gamma$ and $(**)$(iii):

$$M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta.$$

We choose an index h such that $(**)$(iii) holds. It follows that we have the inequality $(\circ)$ and $(\circ\circ)$ as for the assumption $(**)$(ii). This time we set $\Lambda = \Lambda_3 \cup \Lambda_5$ and from $(\circ\circ)$ we deduce that $\Lambda = \varnothing$. (See Table III_2 for $\tilde{\mathscr{D}}$ assumption $(**)$(iii).)

Proof of $(\square)$. If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ and $s = m$, $(\square)$ holds as a consequence of the fact that $r \in H_\gamma$. For the remaining pairs (r, s) (cf. degeneration 2), we consider the identity (6) of Lemma 6.2 $(\circ\circ\circ)$ holds as a consequence of the fact that x is maximum and $\Lambda_3 = \varnothing$. (5.5)(β) follows now from $(**)$(iii), $(\circ\circ\circ)$, and the inequalities

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad M^A_{hs} > M^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

Proposition 5.3 is now proved also under the assumption $(**)$(iii), therefore we may assume

$$M^A_{h'm} > M^B_{h'm} \qquad \text{for every } h' \in H_\beta, \qquad M^A_{h'm} > M^B_{h'm} + 1 \qquad \text{for every } h' \in H_\gamma$$

and the assumption $(**)$ is:

(iv) $$N^A_{hm} = N^B_{hm} \qquad \text{for some } h \in H_\alpha,\ h \neq 1, 2.$$

We choose an index h such that $(**)$(iv) hols. We have:

$(\circ)$ $$N^A_{h\ m-1} - N^A_{h\ m} > N^B_{h\ m-1} - N^B_{h\ m}$$

and from the identity (1) of Lemma 6.1 we deduce:

$(\circ\circ)$

$$|e_{1\ m-1} + e_{2\ m-1} + \cdots + e_{h\ m-1} + f_{0\ m-1} + \cdots + f_{m-2\ m-1} + f_{m-1\ m}|^A > 0.$$

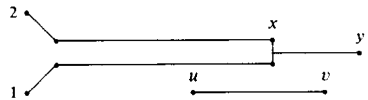
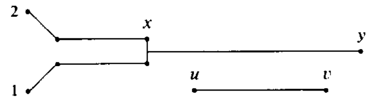
We consider the same sets Λ_τ, $\tau = 0,..., 5$, introduced before (cf. assumption $(**)$(ii)), and the set

$$\Lambda_6 = \{E_{xy} : e^A_{xy} > 0,\ 3 \leqslant x \leqslant u - 1,\ v + 1 \leqslant y \leqslant m - 1\}.$$

We have $\Lambda_0 = \Lambda_6 = \varnothing$, otherwise we go against $(*)$, we set $\Lambda = \bigcup_{\tau=1}^5 \Lambda_\tau$, and from $(\circ\circ)$ it follows that $\Lambda \neq \varnothing$.

If we compare the set Λ relative to the assumption $(**)$(ii) and the one relative to the actual discussion, we see that the sets are the same, therefore Table III_2 for $\tilde{\mathscr{D}}$, assumption $(**)$(iv), is the same as the one for the assumption $(**)$(ii). The proof of $(\square)$ is also the same, with the only

TABLE III_2 for $\tilde{\mathscr{D}}$: Assumption (**)(iii)

Column 1					Column 2
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{L}')$
$\Lambda_3 \neq \varnothing$	$F_{xy} \in \Lambda_3$	1		III	(β) $r \in [v+1, y]$, $s = m$
$\Lambda_3 = \phi$ $\Lambda_5 \neq \varnothing$	$F_y \in \Lambda_5$ x max	2		III	(β) $\begin{cases} r \in [x+1, p], & s \in [v+1, y] \\ r \in [v+1, y], & s = m \end{cases}$

difference that when we consider the pairs $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ such that $s \neq m$ we also have $s > h$ and we have to reproduce only part of the proof.

Proposition 5.3 is now proved also for $(**)$(iv), therefore we may assume: $M^A_{h'm} > M^B_{h'm}$, for every $h' \in H_\beta$; $M^A_{h'm} > M^B_{h'm} + 1$ for every $h' \in H_\gamma$; $N^A_{h'm} > N^B_{h'm}$, $h' \neq 1, 2$, and $(**)$ is:

$$\text{(v)} \qquad N^A_{2m} = N^B_{2m}.$$

We have

$$(\circ) \qquad N^A_{2\ m-1} - N^A_{2\ m} > N^B_{2\ m-1} - N^B_{2\ m} \geqslant 0$$

and from the identity (3) of Lemma 6.1 we deduce

$$(\circ\circ) \qquad [e_{2\ m-1} + f_{0\ m-1} + \cdots + f_{m-2\ m-1}]^A > 0.$$

We use the set Λ_τ, $\tau = 0, 1, \ldots, 5$, already introduced, again $\Lambda_0 = \varnothing$; we set $\Lambda = \bigcup_{\tau=1}^5 \Lambda_\tau$ and $\Lambda \neq \varnothing$. The argument is now the same as the one for the assumption $(**)$(iv). Proposition 5.3 is now completely proved for a degeneration $\mathscr{D}'$ of type III.

7.IV. *The Degenerations* $\mathscr{D}'$ *of Type* IV

We consider first of all the $\mathscr{D}'$ of type IV operating on $F_{p\ m-1} \oplus E_{2b}$, $p < v \leqslant m-2$, such that $B' \leqslant \mathscr{D}'A' < A'$, and by inductive assumption at least one exists. Among them we choose one with v maximum.

Therefore we consider the following pair $\mathscr{D}', \mathscr{D}$ of Table IV:

The assumption $(**)$ now is that one of the following equalities holds for at least an index h:

$$\begin{cases} \text{(i)} & M^A_{hm} = M^B_{hm}, \qquad h \in H_\beta \\ \text{(ii)} & N^A_{2m} = N^B_{2m}. \end{cases}$$

If the assumption $(**)$ is:

$$\text{(i)} \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta,$$

we choose an h minimum, such that (i) holds.

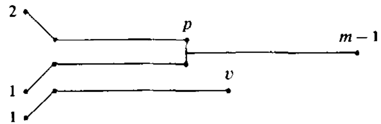
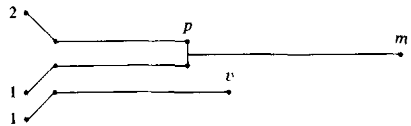
We have:

$$M^A_{h\ m-1} - M^A_{hm} > M^B_{h\ m-1} - M^B_{hm} \geqslant 0$$

as a consequence of $(**)$(i), and the inequality $M^A_{h\ m-1} > M^B_{h\ m-1}$ if $h < m-1$, or $N^A_{2\ m-1} > N^B_{2\ m-1}$ and $N^A_{1\ m-1} \geqslant N^B_{1\ m-1}$ if $h = m-1$. From the identity (5) of Lemma 6.1 and $(\circ)$ we deduce

$$(\circ\circ) \qquad (f_{0,m-1} + \cdots + f_{h-1,m-1})^A = 0.$$

TABLE IV for $(\mathscr{D}', \mathscr{D})$

Degeneration	Factors	Type	Obstruction indices	
$\mathscr{D}'$		IV	$(a,b) \in \mathrm{ob}(\mathscr{D}')$	
$B' \leqslant_{\mathscr{D}'} A' < A'$		(α)	$a \in \{2\} = H_\alpha$,	$b \in [v+1, m-1] = K_\alpha$
v max		(β)	$a \in [p+1, m-1] = H_\beta$,	$b \in [v+1, m-1] = K_\beta$
	$p < v \leqslant m-2$			
$\mathscr{D}$		IV	$(h, \kappa) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$B \nleqslant_{\mathscr{D}} A < A$		(α)	$h = 2$,	$\kappa = m$
		(β)	$h \in H_\beta$,	$\kappa = m$

We define the following sets:

$\Lambda_0 = \{E_{1y} : e^A_{1y} > 0, v + 1 \leqslant y \leqslant m - 2\}.$

$\Lambda_1 = \{E_{2y} : e^A_{2y} > 0, v + 1 \leqslant y \leqslant h - 1\}$

$\Lambda_2 = \{F_{xy} : f^A_{xy} > 0, 0 \leqslant x \leqslant h - 1, v + 1 \leqslant y \leqslant m - 1\}.$

Note that $\Lambda_0 = \varnothing$. In fact if $E_{1y} \in \Lambda_0$, $v + 1 \leqslant y \leqslant m - 2$, we can perform the degeneration $\mathscr{D}'$ of type IV on the factors $F_{p\,m-1} \oplus E_{1y}$, and from the list of the obstruction indices it follows that $B' \leqslant \tilde{\mathscr{D}}'A' < A'$, a contradiction to the choice of the index v.

We set $\Lambda = \Lambda_1 \cup \Lambda_2$ and from ($\circ\circ$) we deduce that $\Lambda \neq \varnothing$. (See Table IV for $\tilde{\mathscr{D}}$ assumption ($**$)(i).)

Explanation of Table IV. We only note that if $\Lambda_1 = \varnothing$ and $\Lambda_2 \neq \varnothing$ then $x \gtreqless v$ by definition. If $x < v$, then we must also have $x < p$, otherwise $\tilde{\mathscr{D}}$ is trivial with respect to $\mathscr{D}'$, against ($*$).

Proof of ($\square$). If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ and $s = m$ (degeneration 1 and 2′), we have

$$M^A_{rm} > M^B_{rm} \qquad \text{as } r < h \text{ and } h \text{ is minimum.}$$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{on}(\mathscr{D}')$ and $s \neq m$ (degeneration 2), we may have $s \geqslant h$ or $s < h$. If $s \geqslant h$ we consider the identity (6) of Lemma 6.2. The contribution in brackets is zero, when evaluated at A, as x is maximum. Therefore ($\circ\circ\circ$) holds and from the identity (6) applied to A and B we get $M^A_{rs} > M^B_{rs}$, as a consequence of the assumption ($**$)(i), and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad M^A_{hs} > M^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

If $s < h$ we consider the identity (7) transformed via the permutation $\sigma = (1, 2)$ of Lemma 6.2. We have ($\circ\circ\circ$) as a consequence of $\Lambda_0 = \Lambda_1 = \varnothing$, and the fact that x is maximum. We deduce, as usual, ($\square$) from the identity 7 and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \quad (B < A); \quad N^A_{1h} \geqslant N^B_{1h} \quad (B < A), \quad N^A_{2s} > N^B_{2s} \quad (s \in K_\alpha).$$

Proposition 5.3 is now proved under the assumption ($**$)(i).

It follows that we may assume

$$M^A_{h'm} > M^B_{h'm} \qquad \text{for every } h' \in H_\beta$$

and ($**$) is now

(ii) $$N^A_{2m} = N^B_{2m}.$$

TABLE IV for $\tilde{\mathscr{D}}$: Assumption (**)(ii)

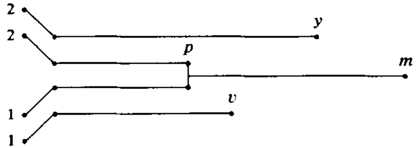
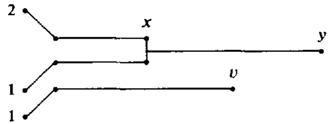
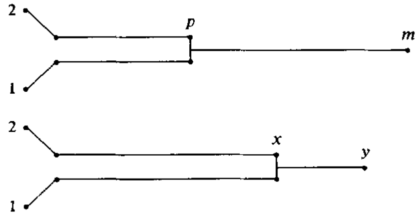
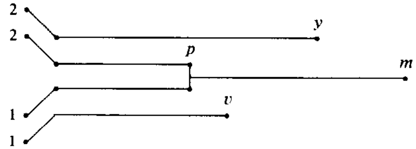
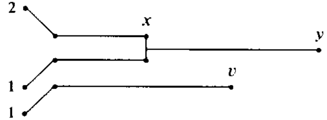
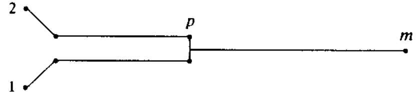
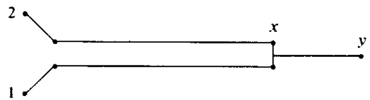
Column 1				Column 2			
Choice in Λ			$\tilde{\mathscr{D}} : B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$		
$\Lambda_1 \neq \varnothing$	$E_{2y} \in \Lambda_1$	1		X	(β)	$r \in [p+1, y]$,	$s = m$
$\Lambda_1 = \varnothing$ $\Lambda_2 \neq \varnothing$	$F_{xy} \in \Lambda_2$ x max	2	 $x < p < v$	IV	(β)	$r \in [x+1, p]$,	$s \in [v+1, y]$
		2′	 $x \geqslant v$	VI	(β)	$r \in [p+1, x]$,	$s = m$

TABLE IV for $\tilde{\mathscr{D}}$: Assumption (**)(ii)

Column 1					Column 2	
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$\Lambda_3 \neq \varnothing$	$E_{2y} \in \Lambda_3$	1		X	(β) $r \in [p+1, y]$,	$s = m$
$\Lambda_3 = \varnothing$ $\Lambda_4 \neq \varnothing$	$F_{xy} \in \Lambda$ x max	2	 $x < p < v$	IV	(β) $r \in [x+1, p]$,	$s = m$
		2′	  $x \geqslant v$	VI	(β) $r \in [p+1, x]$,	$s = m$

We have

$$N^A_{2\ m-1} - N^A_{2\ m} > N^B_{2\ m-1} - N^B_{2\ m} \geqslant 0$$

and from the identity (3) of Lemma 6.1 we deduce:

$$(\circ\circ) \qquad (e_{2\ m-1} + f_{0\ m-1} + f_{3\ m-1} + \cdots + f_{m-2\ m-1})^A > 0.$$

We consider the sets:

$$A_3 = \{E_{2y} : e^A_{2y} > 0, v+1 \leqslant y \leqslant m-2\}$$

$$A_4 = \{F_{xy} : f^A_{xy} > 0, 0 \leqslant x \leqslant m-2, v+1 \leqslant y \leqslant m-1\}.$$

We set $A = A_3 \cup A_4$, and $(\circ\circ)$ implies $A \neq \varnothing$.

Note that Table IV, assumption $(**)$(ii), is essentially the same as Table IV assumption $(**)$(i), but the discussion must be repeated in this case, as we now have different assumptions, namely, (ii) instead of (i). In particular for the pairs $(r, m) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$, $s \neq m$ we consider the identity (8) of Lemma 6.2 $(\circ\circ\circ)$ holds as $A_3 = \varnothing$ and x is maximum.

Therefore we deduce $M^A_{rs} > M^B_{rs}$, as a consequence of $(\circ\circ)$, $(\circ\circ\circ)$, and the inequalities:

$$M^B_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad N^A_{2s} > N^B_{2s} \qquad ((2, s) \in \mathrm{ob}(\mathscr{D}')).$$

Proposition 5.3 is now completely proved for a degeneration $\mathscr{D}'$ of type IV.

7.V. *The Degenerations* $\mathscr{D}'$ *of Type* V

We only have to consider the pairs $\mathscr{D}'$, $\mathscr{D}$ of Table V.

The assumption $(**)$ is:

$$(**) \qquad M^A_{h\ m} = M^B_{h\ m} \qquad \text{for some } h \in H_\beta.$$

We claim that we can reproduce here the same argument of 7.II for the case $q \leqslant m-1$. In fact the degenerations $\tilde{\mathscr{D}}'$ defined in Table II$_1$ never use the indecomposable E_{2q} (for which we don't know a priori here if $e^A_{rq} > 0$ or $e^A_{2q} = 0$); moreover the proof of $(\square)$ for Table II$_1$ never uses the pairs $(2, b) \in \mathrm{ob}(\mathscr{D}')$, $b \in [v+1, q]$.

7.VI. *The Degenerations* $\mathscr{D}'$ *of Type* VI

We only have to consider the pairs $\mathscr{D}'$, $\mathscr{D}$ of Table VI.

The assumption $(**)$ is:

$$(**) \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta.$$

TABLE V for $(\mathscr{L}', \mathscr{L})$

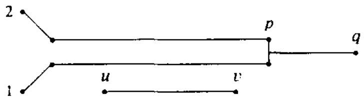
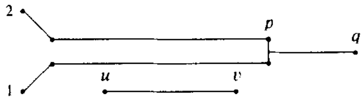
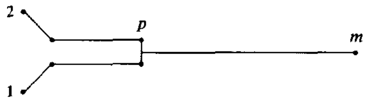
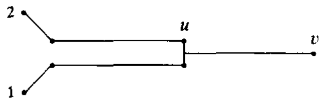
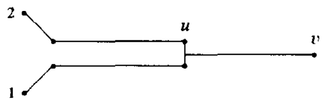
Degeneration	Factors	Type	Obstruction indices
$\mathscr{L}'$ $B' \leqslant_{\mathscr{L}'} A' < A'$	 $q \leqslant m-1$	III	$(a,b) \in \mathrm{ob}(\mathscr{L}')$ (α) $a \in \|0, u-1\| = H_\alpha$, $b \in \|v+1, p\| = K_\alpha$ (β) $a \in \|v+1, p\| = H_\beta$, $b \in \|q+1, m-1\| = K_\beta$
$\mathscr{L}$ $B \nleqslant_{\mathscr{L}} A < A$		III	$(h,\kappa) \in \mathrm{ob}(\mathscr{L}) - \mathrm{ob}(\mathscr{L}')$ (β) $h \in H_\beta$, $\kappa = m$

TABLE VI for $(\mathscr{D}', \mathscr{D})$

Degeneration	Factors	Type	Obstruction indices
$\mathscr{D}'$ $B' \leqslant_{\mathscr{D}'} A' < A'$	 	VI	$(a, b) \in \mathrm{ob}(\mathscr{D}')$ $(\beta)\quad a \in [p+1, u] = H_\beta, \qquad b \in [v+1, m-1] = K_\beta$
$\mathscr{D}$ $B \nleqslant_{\mathscr{D}} A < A$	 	VI	$(h, \kappa) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$ $(\beta)\quad h \in H_\beta, \qquad \kappa = m$

We choose such an index h and we have:

$$(\circ) \qquad M^A_{h\ m-1} - M^A_{h\ m} > M^B_{h\ m-1} - M^B_{h\ m}$$

as a consequence of $(**)$ and the fact that $(h, m-1) \in \mathrm{ob}(\mathscr{D}')$.
From the identity (5) of Lemma 6.1 and $(\circ)$ we deduce:

$$(\circ\circ) \qquad (f_{0\ m-1} + \cdots + f_{h-1\ m-1})^A > 0.$$

We set $\Lambda = \{F_{xy} : f^A_{xy} > 0,\ 0 \leqslant x \leqslant h-1,\ v+1 \leqslant y \leqslant m-1\}$ and $(\circ\circ)$ says that $\Lambda \neq \varnothing$. (See Table VI for $\tilde{\mathscr{D}}$.)

Explanation of Table VI. We only note that all the factors $F_{xy} \in \Lambda$ must be such that $x < p$, otherwise the degeneration $\tilde{\mathscr{D}}$ is trivial with respect to $\mathscr{D}'$, against $(*)$.

Proof of $(\square)$. For any $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ we consider the identity (6) of Lemma 6.2; $(\circ\circ\circ)$ holds as x is maximum.
As usual we deduce $(\square)$ from $(**)$, $(\circ\circ\circ)$, and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad M^A_{hs} > M^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

7.VII. *The Degenerations* $\mathscr{D}'$ *of Type* VII

We only have to consider the pairs $\mathscr{D}'$, $\mathscr{D}$ of Table VII.

The assumption $(**)$ is:

$$(**) \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta.$$

We choose such an h and we have:

$$(\circ) \qquad M^A_{h\ m-1} - M^A_{hm} > M^B_{h\ m-1} - M^B_{hm} \geqslant 0$$

as a consequence of $(**)$ and the fact that $(h, m-1) \in \mathrm{ob}(\mathscr{D}')$.
From the identity (5) of Lemma 6.1 and $(\circ)$ we deduce

$$(\circ\circ) \qquad (f_{0\ m-1} + \cdots + f_{h-1\ m-1})^A > 0.$$

We consider the following set:

$$\Theta = \{F_{cd} : f^A_{cd} > 0, 0 \leqslant c \leqslant h-1, v+1 \leqslant d \leqslant m-1\}$$

which is non-empty, as a consequence of $(\circ\circ)$. We choose $F_{xy} \in \Theta$ with the property that x is maximum. Note that $x < p$, otherwise we could perform the degeneration $\mathscr{D}^*$ of type VII on the factors $E_{xy} \oplus E_{1v}$, which is trivial, against $(*)$.

TABLE VI for $\tilde{\mathscr{D}}$

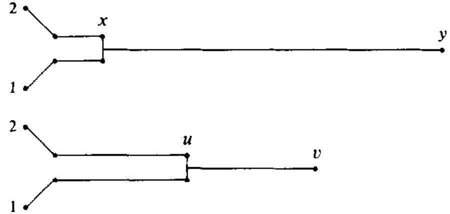
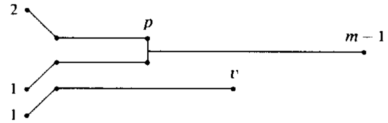
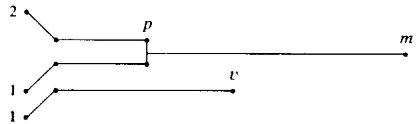
Column 1			Column 2
Choice in Λ	$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$
$F_{xy} \in \Lambda$ x max $(x < p)$		VI	(β) $r \in [x+1, p]$, $s \in [v+1, y]$

TABLE VII for $(\mathscr{L}', \mathscr{L})$

Degeneration	Factors	Type	Obstruction indices
$\mathscr{L}'$ $B' \leqslant \mathscr{L}'A' < A'$		VII	$(a, b) \in \mathrm{ob}(\mathscr{L}')$ (α) $a \in \{1\} = H_\alpha$, $b \in [p+1, v] = K_\alpha$ (β) $a \in [p+1, v] = H_\beta$, $b \in [p+1, m-1] = K_\beta$
$\mathscr{L}$ $B \nleqslant \mathscr{L}A < A$		VII	$(h, \kappa) \in \mathrm{ob}(\mathscr{L}) - \mathrm{ob}(\mathscr{L}')$ (β) $h \in H_\beta$, $\kappa = m$

Next we consider the set

$$\Lambda_1 = \{E_{2w} : x < w \leqslant h-1\}$$

and we set:

$$\Lambda = \Lambda_1 \cap \{F_{xy}\}.$$

Clearly $\Lambda \neq \varnothing$. For $\tilde{\mathscr{D}}$ see Table VII.

Explanation of Table VII. Note that if $\Lambda_1 \neq \varnothing$ and w is maximum, then $w \leqslant h-1 < v$, but we may have $w < p$ or $w \geqslant p$. If $w < p$ in column 2 we do not have the pairs $(1, s)$, $s \in [w+1, p]$.

Proof of ($\square$). If $(1, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$, we consider the identity (9) of Lemma 6.2 transformed via the permutation $\sigma = (1, 2)$. ($\circ\circ\circ$) holds, in fact:

$(e_{2s} + \cdots + e_{2\,h-1})^{\Lambda} = 0$ in 1 as w is maximum; in 2 as $\Lambda_1 \neq \varnothing$;

$((f_{s\,s+1} + \cdots + f_{s\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1}))^{\Lambda} = 0$
in 1 as x is maximum and $x < w$; in 2 as x is maximum.

We use the same identity for A and B and (5.5)(α) follows as a consequence of ($**$), ($\circ\circ\circ$), and the inequalities:

$$M_{sm}^A \geqslant M_{sm}^B \qquad (B < A); \qquad N_{1h}^A > N_{1h}^B \qquad ((1, h) \in \mathrm{ob}(\mathscr{D}')).$$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $s > h$ we consider the identity (6) of Lemma 6.2. ($\circ\circ\circ$) holds, if fact:

$((f_{rs} + \cdots + f_{h-1\,s}) + \cdots + (f_{r\,m-1} + \cdots + f_{h-1\,m-1}))^{\Lambda} = 0$ in 1 as x is maximum ($x < w$); in 2 as x is maximum.

From the same identity we deduce (5.5)(β), as a consequence of ($**$), ($\circ\circ\circ$), and the inequalities:

$$M_{rm}^A \geqslant M_{rm}^B \qquad (B < A); \qquad M_{hs}^A > M_{hs}^B \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

If $s \leqslant h$ we use the identity (7) (where we permute the indices 1, 2). ($\circ\circ\circ$) holds, in fact:

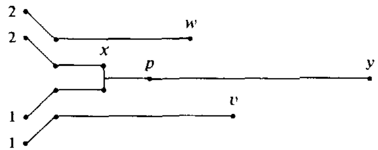
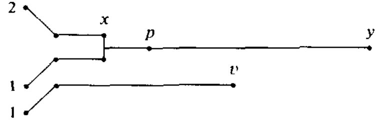
$(e_{2s} + \cdots + e_{2\,h-1})^{\Lambda} = 0$ in 1 as w is maximum; in 2 as $\Lambda_1 \neq \varnothing$;

$(f_{rs} + \cdots + f_{r\,m-1} + \cdots + f_{h-1\,s} + \cdots + f_{h-1\,m-1})^{\Lambda} = 0$
in 1 as x is maximum ($x < w$); in 2 as x is maximum.

(5.5)(β) is now a consequence of ($**$), ($\circ\circ\circ$), and the inequalities:

$$M_{rm}^A \geqslant M_{rm}^B \quad (B < A); \quad N_{2s}^A \geqslant N_{2s}^B \quad (B < A); \quad N_{1h}^A > N_{1h}^B \quad ((1, h) \in \mathrm{ob}(\mathscr{D}')).$$

TABLE VII for $\tilde{\mathscr{D}}$

Column 1				Column 2			
Choice in Λ			$\tilde{\mathscr{D}} : B \leqslant \tilde{\mathscr{D}} A < A$	Type		$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{L}')$	
$\Lambda_1 \neq \varnothing$	$E_{2w} \in \Lambda_1$ w max	1		X	(α) (β)	$r = 1,$ $r \in [x + 1, p],$	$s \in [w + 1, p](w > p)$ $s \in [w + 1, y]$
$\Lambda_1 = \varnothing$	F_{xy}	2		VII	(α) (β)	$r = 1,$ $r \in [x + 1, p],$	$s \in [x + , p]$ $s \in [x + 2, y]$

7.VIII. *The Degenerations* $\mathscr{D}'$ *of Type* VIII

We consider first all the $\mathscr{D}'$ of type VIII operating on the factors $F_{uv} \oplus F_{p\,m-1}$ $(0 \leqslant u < p)$, such that $B' \leqslant \mathscr{D}'A' < A'$ and by the inductive assumption at least one of them exists. We fix the index p and we choose the index u maximum possible, with respect to the condition $0 \leqslant u < p$, therefore we have Table VIII for $(\mathscr{D}', \mathscr{D})$.

$$(**) \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H''_\beta,$$

and we choose such as index h.

The choice of the index u maximum implies that it is the following empty set:

$$\Lambda_0 = \{F_{xy} : f^A_{xy} > 0, u+1 \leqslant x \leqslant p-1, p < y \leqslant v\} = \varnothing.$$

Moreover, not to go against the assumption $(*)$, we also have that the following sets are empty:

$$\Lambda_1 = \{F_{xy} : f^A_{xy} > 0, p \leqslant x < y < v\} = \varnothing$$

$$\Lambda_2 = \{F_{xy} : f^A_{xy} > 0, p \leqslant x \leqslant h-1; v+1 \leqslant y \leqslant m-1\} = \varnothing.$$

In fact if we have a factor F_{xy} of A, $F_{xy} \in \Lambda_1$ (resp. Λ_2) we could perform the degeneration $\mathscr{D}^*$ of type VI (resp. VIII) on the factors $F_{uv} \oplus F_{xy}$, which is trivial with respect to $\mathscr{D}'$.

We consider the following sets:

$$\Lambda_3 = \{F_{xy} : f^A_{xy} > 0, u+1 \leqslant x \leqslant p-1, v+1 \leqslant y \leqslant m-1\}$$

$$\Lambda_4 = \{F_{ij} : f^A_{ij} > 0, u+1 \leqslant i < j \leqslant p-1\}$$

$$\Lambda_5 = \{E_{1w} : e^A_{1w} > 0, u+1 \leqslant w \leqslant h-1\}$$

$$\Lambda_6 = \{E_{2z} : e^A_{2z} > 0, u+1 \leqslant z \leqslant h-1\}.$$

We set $\Lambda = \bigcup_{\tau=3}^{6} \Lambda_\tau \cup \{F_{uv}\} \neq \varnothing$. See Table VIII for $\tilde{\mathscr{D}}$.

Explanation of Table VIII. If $\Lambda_\tau \neq \varnothing$, $\tau = 3, 4, 5, 6$, we choose $F_{xy} \in \Lambda_3$ with x maximum, $F_{ij} \in \Lambda_4$ with j maximum, $F_{1w} \in \Lambda_5$ with w maximum, and $E_{2z} \in \Lambda_6$ with z maximum. Clearly if one of the previous sets is empty the corresponding integer is not defined. If $\Lambda_5 \neq \varnothing$ and $\Lambda_6 \neq \varnothing$ we may assume that $z \leqslant w$ (otherwise we use the permutation $\sigma = (1, 2)$), i.e., we substitute the factor E_{1w} with E_{2z}, For the same reason we may also assume that if $\Lambda_5 = \varnothing$ also $\Lambda_6 = \varnothing$.

If $j \geqslant \max(x, w)$ (or if $\Lambda_3 = \varnothing$ and $j \geqslant w$, or if $\Lambda_5 = \varnothing$ and $j \geqslant x$, or if $\Lambda_3 \cup \Lambda_5 = \varnothing$), we perform the degeneration 1.

TABLE VIII for $(\mathscr{D}', \mathscr{D})$

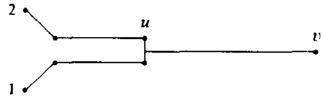
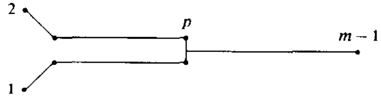
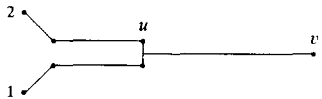
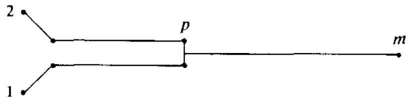
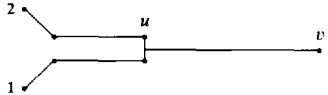
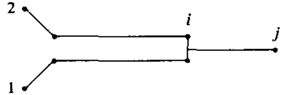
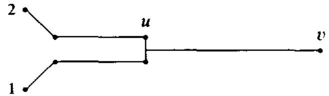
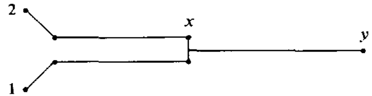
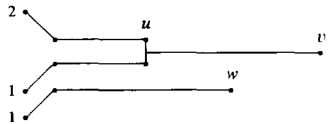
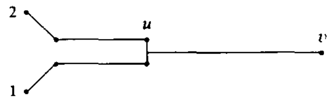
Degeneration	Factors	Type	Obstruction indices
$\mathscr{D}'$ $B' \leqslant \mathscr{D}'A' < A'$ u max	 	VIII	$(a, b) \in \mathrm{ob}(\mathscr{D}')$ (α) $a \in \{1, 2\} = H_\alpha$, $b \in [p+1, v] = K_\alpha$ (β) $\begin{cases} a \in [u+1, p] = H'_\beta, & b \in [p+1, v] = K'_\beta \\ a \in [p+1, v] = H''_\beta, & b \in [v+1, m-1] = K''_\beta \end{cases}$ (γ) $a \in [b+1, v-1] = H_\gamma$, $b \in [p+2, v] = K_\gamma$
$\mathscr{D}$ $B \nleqslant \mathscr{D}A < A$	 	VIII	$(h, k) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$ (β) $h \in H''_\beta$, $\kappa = m$

TABLE VIII for $\mathscr{D}$

Column 1					Column 2		
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$		
$\bigcup_{\tau=3}^{6} \Lambda_\tau \neq \varnothing$	$F_{xy} \in \Lambda_3$ x max $F_{ij} \in \Lambda_4$ j max $E_{1w} \in \Lambda_5$ w max $E_{2z} \in \Lambda_3$ z max $z \leqslant w$	1	  $j \leqslant \max(x, w)$	VI	(β)	$r \in [u+1, i]$,	$s \in [j+1, p]$
		$1'$	  $x \geqslant \max(j, w)$	VIII	(α) (β) (γ) 	$r \in \{1, 2\}$, $r \in [u+1, x]$, $r \in [x+1, p]$, $r \in [x+1, p-1]$, $r \in [x+1, p]$,	$s \in [x+1, p]$ $s \in [x+1, p]$ $s \in [v+1, y]$ $s \in [x+2, p]$ $s \in [p+1, v]$
		$1''$	 $w \geqslant \max(j, x)$ $(w < p)$	IV	(α) (β)	$r = 2$, $r \in [u+1, p-1]$,	$s \in [w+1, p]$ $s \in [w+1, p]$
$\bigcup_{\tau=3}^{6} \Lambda_\tau = \varnothing$		2		IX $+\sigma$	(α) (β)	$r = 2$, $r \in [u+1, p-1]$,	$s \in [u+1, p]$ $s \in [u+2, p]$

If $x \geqslant \max(j, w)$ (or if $A_4 = \varnothing$ and $x \geqslant w$, or if $A_5 = \varnothing$ and $x \geqslant j$, or if $A_4 \cup A_5 = \varnothing$), we perform the degeneration 1'.

If $w \geqslant \max(x, j)$ (or if $A_3 = \varnothing$ and $w \geqslant j$, or if $A_4 = \varnothing$ and $w \geqslant x$, or if $A_3 \cup A_4 = \varnothing$), we perform the degeneration 1".

If $\bigcup_{\tau=3}^{6} A_\tau = \varnothing$ we perform the degeneration 2.

Proof of ($\square$). If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (α) (degenerations 1', 1", and 2) we consider the identity (9) of Lemma 6.2 (or the same identity transformed via the permutation $\sigma = (1, 2)$ if $r = 1$). ($\circ\circ\circ$) holds. In fact if $r = 2$:

$(e_{1s} + \cdots + e_{1\,h-1})^A = 0$ in 1' as w is max $(w \leqslant x < s)$;
in 1" as w is max; in 2 as $A_5 = \varnothing$.

If $r = 1$:

$(e_{2s} + \cdots + e_{2\,h-1})^A = 0$ in 1' as z is max $(z \leqslant w \leqslant x < s)$.

Moreover,

$((f_{s\,s+1} + \cdots + f_{s\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1}))^A = 0$

in 1' as j is max $(j \leqslant x < s)$; $A_0 = \varnothing$; x is max; $A_2 = \varnothing$;

in 1" as j is max $(j \leqslant w < s)$; $A_0 = \varnothing$; x is max $(x \leqslant w < s)$; $A_1 = \varnothing$, $A_2 = \varnothing$;

in 2 as $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \varnothing$.

(5.3)(α) is now a consequence of the inequalities:

$$M^A_{sm} \geqslant M^B_{sm} \qquad (B < A); \qquad N^A_{1h} > N^B_{1h} \qquad ((1, h) \in \mathrm{ob}(\mathscr{D}')).$$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $r \in [x+1, b]$, $s \in [v+1, y]$ (degeneration 1'), we consider the identity (6) of Lemma 6.2 ($\circ\circ\circ$) holds, in fact:

$[(f_{rs} + \cdots + f_{h-1\,s}) + \cdots + (f_{r\,m-1} + \cdots + f_{h-1\,m-1})]^A = 0$ as x is max and $A_2 = \varnothing$ and 5.3(β) follows from ($\circ\circ\circ$), from ($**$), and from the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad M^A_{hs} > M^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

For all the remaining pairs $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathbb{D}')$ of type (β) we consider the identity (7) of Lemma 6.2. ($\circ\circ\circ$) holds, in fact:

$(e_{1s} + \cdots + e_{1\,h-1})^A = 0$ in 1 as w is max $(w \leqslant j < s)$; in 1' as w is max;

$(w \leqslant x < s)$; in 1" as w is max; in 2 as $A_5 = \varnothing$;

$((f_{rs} + \cdots + f_{r\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1}))^A = 0$

in 1 as j is max; $\Lambda_0 = \emptyset$; x is max $(x \leqslant j < s)$; $\Lambda_1 = \emptyset$, $\Lambda_2 = \emptyset$;

in 1′ as j is max $(j \leqslant x < s)$; $\Lambda_0 = \emptyset$; x is max; $\Lambda_1 = \emptyset$, $\Lambda_2 = \emptyset$;

in 1″ as j is max $(j \leqslant w < s)$; $\Lambda_0 = \emptyset$; x is max $(x \leqslant w < s)$; $\Lambda_1 = \emptyset$, $\Lambda_2 = \emptyset$;

in 2 as $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4 = \emptyset$.

We deduce (5.3)(β) from ($\circ\circ\circ$), ($**$), and from the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad N^A_{2h} > N^B_{2h} \qquad ((2, h) \in \mathrm{ob}(\mathscr{D}')).$$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (γ) (degeneration 1′) and $h \geqslant s$ (i.e., if $(r, s) \in [x+1,\ p-1] \times [x+2, p]$, or if $(r, s) \in [x+1, p] \times [p+1, v]$ and $h \geqslant s$) we consider the identity (7) of Lemma 6.2. ($\circ\circ\circ$) holds, in fact:

$(e_{1s} + \cdots + e_{1\,h-1})^{\Lambda} = 0$ as w is max $(w \leqslant x < s)$;

$((f_{rs} + \cdots + f_{r\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1}))^{\Lambda} = 0$, as $\Lambda_0 = \emptyset$, x is max, $\Lambda_2 = \emptyset$, and if $x + 1 \leqslant r < s \leqslant p$, j is max $(j \leqslant x < s)$.

(5.3)(γ) now follows from the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \quad (B < A); \quad N^A_{2h} > N^B_{2h} \quad ((2, h) \in \mathrm{ob}(\mathscr{D}')); \quad N^A_{1s} > N^B_{1s}$$

(proved before as $(1, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (α)).

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (γ) and $h < s$ we consider the identity (6) of Lemma 6.2. ($\circ\circ\circ$) holds, as $\Lambda_0 = \emptyset$, x is max, $\Lambda_1 = \Lambda_2 = \emptyset$. (5.3)($\gamma$) follows from the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad M^A_{hs} > M^B_{hs} + 1 \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

7.IX. *The Degenerations* $\mathscr{D}'$ *of Type* IX

We only have to consider the pairs $(\mathscr{D}', \mathscr{D})$ of Table IX.

The assumption ($**$) is now that one of the following equalities holds for at least an index h:

$$(**) \qquad \begin{cases} \text{(i)} & M^A_{h\,m} = M^B_{h\,m}, \qquad h \in H_\beta \\ \text{(ii)} & N^A_{1\,m} = N^B_{1\,m}. \end{cases}$$

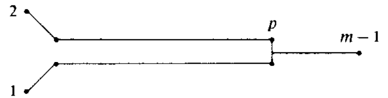
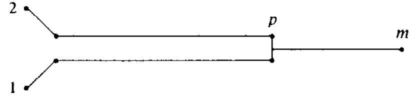
If the assumption ($**$) is:

$$\text{(i)} \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta$$

we choose such an index h. We have

$$(\circ) \qquad M^A_{h\,m-1} - M^A_{hm} > M^B_{h\,m-1} - M^B_{hm} \geqslant 0$$

TABLE IX for $(\mathscr{L}', \mathscr{L})$

Degeneration	Factors	Type	Obstruction indices	
$\mathscr{L}'$		IX	$(a, b) \in \mathrm{ob}(\mathscr{L}')$	
$B' \leqslant \mathscr{L}'A' < A'$		(α)	$a \in \{1\} = H_\alpha$,	$b \in \lvert p+1, m-1 \rvert = K_\alpha$
		(β)	$a \in \lvert p+1, m-1 \rvert = H_\beta$,	$b \in \lvert p+2, m-1 \rvert = K_\beta$
$\mathscr{L}$		IX	$(h, \kappa) \in \mathrm{ob}(\mathscr{L}) - \mathrm{ob}(\mathscr{L}')$	
$B \nleqslant \mathscr{L}A < A$		(α)	$h = 1$,	$\kappa = m$
		(β)	$h \in H_\beta$,	$\kappa = m$

and from the identity (5) of Lemma 6.1 we deduce:

$$(\circ\circ) \qquad [f_{0\ m-1} + f_{3\ m-1} + \cdots + f_{h-1\ m-1}]^A > 0.$$

We consider the sets:

$$\Lambda_0 = \{F_{ij} : f_{ij}^A > 0,\ p \leqslant i < j \leqslant m-1\}$$

$$\Lambda_1 = \{F_{xy} : f_{xy}^A > 0; 0 \leqslant x \leqslant p-1; p+1 \leqslant y \leqslant m-1\}.$$

Note that $\Lambda_0 = \varnothing$, otherwise we can perform $\mathscr{D}^*$ of type IX on a factor $F_{xy} \in \Lambda_0$, against $(*)$. It follows that $\Lambda_1 \neq \varnothing$, and we choose once for ever a factor $E_{xy} \in \Lambda_1$, such that x is maximum.

Next we define the following sets:

$$\Lambda_2 = \{E_{2z} : e_{2z}^A > 0, x+1 \leqslant z \leqslant m-2\}$$

$$\Lambda_3 = \{F_{vw} : f_{vw}^A > 0, x+1 \leqslant v < w \leqslant p\}.$$

We set $\Lambda = (F_{xy}\} \cup \Lambda_2 \cup \Lambda_3$, and we have $\Lambda \neq \varnothing$. (See Table IX for $\tilde{\mathscr{D}}$ assumption $(**)$(i).)

Proof of ($\square$). If $(1, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ we consider the identity (9) of Lemma 6.2 transformed via the permutation $\sigma = (1, 2)$. $(\circ\circ\circ)$ holds, in fact:

$(e_{2s} + \cdots + e_{2\ h-1})^A = 0$ in 1 as $\Lambda_2 = \varnothing$; in 2′ as z in max;

$((f_{s\ s+1} + \cdots + f_{s\ m-1}) + \cdots + (f_{h-1\ h} + \cdots + f_{h-1\ m-1}))^A = 0$ in 1 as $\Lambda_3 = \varnothing$, $\Lambda_0 = \varnothing$, x is max; in 2′ as w is max $(x < z < s;\ w < z < s+1)$; x is max and $\Lambda_0 = \varnothing$.

(5.5)(α) follows now from $(**)$(i), $(\circ\circ\circ)$, and the inequalities:

$$M_{sm}^A \geqslant M_{sm}^B \qquad (B < A); \qquad N_{1h}^A > N_{1h}^B \qquad ((1, h) \in \mathrm{ob}(\mathscr{D}')).$$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $h < s$, we consider the identity (6) of Lemma 6.2. $(\circ\circ\circ)$ holds, in fact:

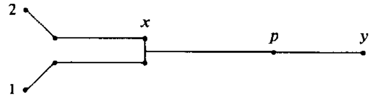
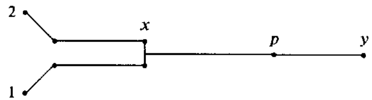
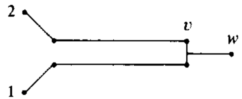
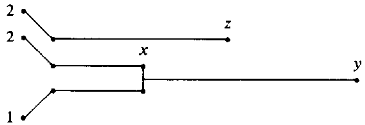
$$[(f_{rs} + \cdots + f_{h-1\ s}) + \cdots + (f_{r\ m-1} + \cdots + f_{h-1\ m-1})]^A = 0$$

in 1 as $\Lambda_3 = \varnothing$, x is max and $\Lambda_0 = \varnothing$; in 2 and 2′ as w is max, x is max and $\Lambda_0 = \varnothing$. From the same identity we deduce (5.5)(β), as a consequence of $(**)$(i), $(\circ\circ\circ)$, and the inequalities:

$$M_{rm}^A \geqslant M_{rm}^B \qquad (B < A); \qquad M_{hs}^A > M_{hs}^B \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $h \geqslant s$, we consider the identity

TABLE IX for $\tilde{\mathscr{D}}$: Assumption (**)(i)

Column 1				Column 2	
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$
$\Lambda_2 \cup \Lambda_3 = \varnothing$	$F_{xy} \in \Lambda$	1		IX	(α) $r = 1$, $s \in [x+1, p]$ (β) $r \in [x+1, p]$, $s \in [x+2, y]$
$\Lambda_2 \cup \Lambda_3 \neq \varnothing$	$Z_{2z} \in \Lambda_2$ z max $F_{vw} \in \Lambda_3$ w max	2	  $z \leqslant w$ (or $\Lambda_2 = \varnothing$)	VI	(β) $r \in [x+1, v]$, $s \in [w+1, y]$
		2′	 $z > w$ (or $\Lambda_3 = \varnothing$)	IV	(α) $r = 1$, $s \in [z+1, p]$ (β) $r \in [x+1, p]$, $s \in [z+1, y]$

(7) of Lemma 6.2 transformed via the permutation $\sigma = (1, 2)$. ($\circ\circ\circ$) holds, in fact:

$(e_{2s} + \cdots + e_{2\,h-1})^A = 0$ in 1 as $A_2 = \varnothing$; in 2 and 2′ as z is max;

$((f_{rs} + f_{r\,m-1}) + \cdots + (f_{h-1\,h} + \cdots + f_{h-1\,m-1}))^A = 0$, in 1 as $A_3 = \varnothing$, x is max and $A_0 = \varnothing$; in 2 and 2′ as w is max, x is max and $A_0 = \varnothing$.

Proposition 5.3 is now proved under the assumption (**)(i). Therefore we may assume:

$$M^A_{hm} > M^B_{hm} \qquad \text{for every } h \in H_\beta,$$

and the assumption (**) is now:

$$\text{(ii)} \qquad N^A_{1m} = N^B_{1m}.$$

We have:

$$N^A_{1\,m-1} - N^A_{1m} > N^B_{1\,m-1} - N^B_{1m} \geqslant 0$$

and from the identity (3) of Lemma 6.1 we deduce:

$$(\circ\circ) \qquad |e_{1\,m-1} + f_{0\,m-1} + \cdots + f_{m-2\,m-1}|^A > 0.$$

We consider the sets A_0 and A_1 already defined and the set:

$$A_4 = \{E_{1y} : e^A_{1y} > 0,\ p + 1 \leqslant y \leqslant m - 1\}.$$

We set $A = A_1 \cup A_4$, and, as we have proved that $A_0 = \varnothing$ from ($\circ\circ$), we deduce $A \neq \varnothing$. (See Table IX for $\tilde{\mathscr{D}}$.)

Proof of ($\square$). If $(2, s) \in \text{ob}(\tilde{\mathscr{D}}) - \text{ob}(\mathscr{D}')$ type (α)), we consider the identity (10) of Lemma 6.2. ($\circ\circ\circ$) holds, in fact:

$(e_{1s} + \cdots + e_{1\,m-1})^A = 0$ as $A_4 = \varnothing$;

$((f_{s\,s+1} + \cdots + f_{s\,m-1}) + \cdots + f_{m-2\,m-1}))^A = 0$ as $A_0 = \varnothing$.

If $(r, s) \in \text{ob}(\tilde{\mathscr{D}}) - \text{ob}(\mathscr{D}')$ is of type (β) and $s = m$, ($\square$) holds, as $r \in H_\beta$. For the remaining pairs (r, s) of type (β) (degeneration 2), we consider the identity (8) of Lemma 6.2. ($\circ\circ\circ$) holds, in fact:

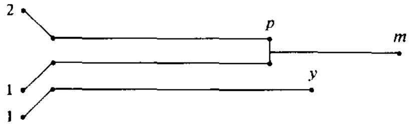
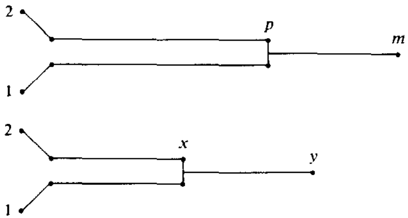
$(e_{1s} + \cdots + e_{1\,m-1})^A = 0$ as $A_4 = \varnothing$;

$((f_{rs} + \cdots + f_{r\,m-1}) + \cdots + f_{m-2\,m-1})^A = 0$, as x is max and $A_0 = \varnothing$.

From the same identity we deduce (5.5)(β), as a consequence of (**)(ii), ($\circ\circ\circ$), and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \quad (B < A); \qquad N^A_{1s} > N^B_{1s} \quad ((1, s) \in \text{ob}(\mathscr{D}')).$$

TABLE IX for $\tilde{\mathscr{D}}$: Assumption (**)(ii)

Column 1				Column 2		
Choice in Λ			$\tilde{\mathscr{D}} : B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$\Lambda_4 \neq \varnothing$	$E_{1y} \in \Lambda_4$	1		VII (β)	$r \in [p+1, m-1]$,	$s = m$
$\Lambda_4 = \varnothing$	$F_{xy} \in \Lambda_1$ x max	2		VIII (α)	$r = 2$,	$s = m$
				(β)	$\begin{cases} r \in [x+1, p], \\ r \in [p+1, y], \end{cases}$	$s \in [p+1, y]$ $s = m$
				(γ)	$r \in [p+1, y-1]$,	$s \in [p+2, y]$

If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (γ) we consider the same identity (8). $(\circ\circ\circ)$ holds this time simply because $\Delta_0 \cup \Delta_4 = \varnothing$, and (5.5)($\gamma$) follows from $(**)$(ii), $(\circ\circ\circ)$, and the inequalities:

$$M^A_{rm} > M^B_{rm} \qquad (r \neq H_\beta); \qquad N^A_{1s} > N^B_{1s} \qquad ((1, s), \mathrm{ob}(\mathscr{D}')).$$

7.X. *The Degenerations $\mathscr{D}'$ of Type* X.

We consider all the $\mathscr{D}'$'s of type X, operating on $E_{2v} \oplus E_{p\, m-1} \oplus E_{1u}$ $(p < u \leqslant v)$, such that $B' \leqslant \mathscr{D}'A' < A'$ and by the inductive assumption at least one exists. Among them we choose one with u maximum.

We have the pairs $(\mathscr{D}', \tilde{\mathscr{D}})$ of Table X.

The assumption $(**)$ is:

$$(**) \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta$$

and we choose such an index h. We have:

$$(\circ) \qquad M^A_{h\, m-1} - M^A_{hm} > M^B_{h\, m-1} - M^B_{hm}$$

and from the identity (5) of Lemma 6.1 it follows:

$$(\circ\circ) \qquad (f_{0\, m-1} + \cdots + f_{h-1\, m-1})^A > 0.$$

We consider the sets

$$\Delta_0 = \{F_{xy} : f^A_{xy} > 0,\ p \leqslant x \leqslant v-1,\ v+1 \leqslant y \leqslant m-1\}$$

$$\Delta_1 = \{F_{xy} : f^A_{xy} > 0,\ 0 \leqslant x \leqslant p-1,\ v+1 \leqslant y \leqslant m-1\}.$$

Note that $\Delta_0 = \varnothing$. In fact if $F_{xy} \in \Delta_0 \neq \varnothing$, and $p \leqslant x \leqslant u-1$, we could perform the degeneration $\mathscr{D}^*$ of type X on the factors $E_{2v} \oplus F_{xy} \oplus E_{1u}$, if $u \leqslant x \leqslant v-1$ we could perform the degeneration $\mathscr{D}^*$ of type VII of the factors $E_{2v} \oplus F_{xy}$, and in both cases $\mathscr{D}^*$ is trivial with respect to $\mathscr{D}'$, against $(*)$.

We set $\Delta = \Delta_1$ and from $(\circ\circ)$ we deduce $\Delta \neq \varnothing$. (See Table X for $\tilde{\mathscr{D}}$.)

Proof of ($\square$). If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ and $h < s$ we consider the identity (6) of Lemma 6.2. $(\circ\circ\circ)$ holds as x is max and $\Delta_0 = \varnothing$. (5.5)(β) follows from $(\circ\circ\circ)$, $(**)$, and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad M^A_{hs} > M^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

If $h \geqslant s$ we consider the identity (7). $(\circ\circ\circ)$ holds, in fact:

$(e_{1s} + \cdots + e_{1\, h-1})^A = 0$ as u is max;

$((f_{rs} + \cdots + f_{r\, m-1}) + \cdots + (f_{h-1\, h} + \cdots + f_{h-1\, m-1}))^A = 0$ as x is max and $\Delta_0 = \varnothing$.

TABLE X for $(\mathscr{D}', \mathscr{D})$

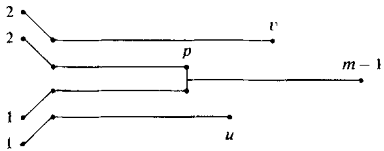
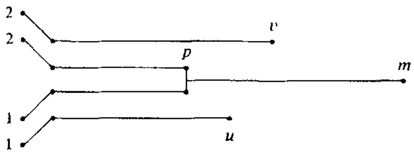
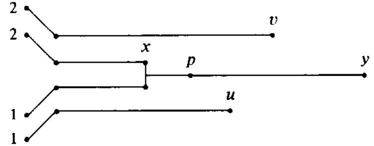
Degeneration	Factors	Type	Obstruction indices
$\mathscr{D}'$ $B' \leqslant_{\mathscr{D}'} A' < A'$	 	X	$(a, b) \in \mathrm{ob}(\mathscr{D}')$ (α) $a \in \{2\} = H_\alpha$, $b \in [u+1, v] = K_\alpha$ (β) $a \in [p+1, v] = H_\beta$, $b \in [u+1, m-1] = K_\beta$
$\mathscr{D}$ $B \nleqslant_{\mathscr{D}} A < A$	 	X	$(h, \kappa) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$ (β) $h \in H_\beta$, $\kappa = m$

TABLE X for $\tilde{\mathscr{D}}$

Column 1				Column 2
Choice in Λ		$\tilde{\mathscr{D}} : B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{L}')$
$\Lambda_1 \neq \varnothing$	$F_{xy} \in \Lambda_1$ x max		X	(β) $r \in [x+1, p]$, $s \in [u+1, y]$

(5.5)(β) now follows from ($\circ\circ\circ$), ($**$), and the inequalities:

$$M^A_{rm} \geqslant M^B_{rm}; \qquad N^A_{1s} \geqslant N^B_{1s} \qquad (B < A); \qquad N^A_{2h} > N^B_{2h}$$
$$((u+1 \leqslant s \leqslant y, (2,h) \in \mathrm{ob}(\mathscr{D}')).$$

7.XI. *The Degenerations* $\mathscr{D}'$ *of Type* XI

We consider first the pairs $(\mathscr{D}', \mathscr{D})$ of Table XI_1. The assumption ($**$) is:

$$(**) \qquad M^A_{hm} = M^B_{hm} \qquad \text{for some } h \in H_\beta.$$

If is easy to check that we can reproduce the same argument as in 7.II which gives rise to Table II_1. In fact the assumption ($**$) is the same, and the degeneration $\tilde{\mathscr{D}}$ is defined in terms of the factor E_{uv}, moreover, even if the set $\mathrm{ob}(\mathscr{D}')$ in 7.II properly contains the set $\mathrm{ob}(\mathscr{D}')$ we have here, the pairs $(a, b) \in \mathrm{ob}(\mathscr{D}')_{7.\mathrm{II}} - \mathrm{ob}(\mathscr{D}')_{7.\mathrm{XI}}$ where never used in the proof of ($\square$) (cf. 7.II).

We may now assume that the only degenerations $\mathscr{D}'$ of type XI and such that $B' \leqslant \mathscr{D}'A' < A'$ act on the following factors of A': $E_{2\,m-1} \oplus E_{uv} \oplus E_{1p}$, $p \leqslant m-1$. The only liftings $\mathscr{D}$ which are not trivial with respect to $\mathscr{D}'$ are again of type XI, and, up to permutation, either act on the factors (of A) $E_{2m} \oplus E_{uv} \oplus E_{1p}$, $p \leqslant m-1$, or on $E_{2m} \oplus E_{uv} \oplus E_{1m}$ (which can occur if $p = m-1$).

We consider next the pairs $(\mathscr{D}', \mathscr{D})$ of Table XI_2. The assumption ($**$) is now that one of the following equalities holds for at least an index h:

$$(**) \qquad \begin{array}{lll} \text{(i)} & M^A_{hm} = M^B_{hm}, & h \in H_\beta \\ \text{(ii)} & N^A_{hm} = N^B_{hm}, & h \in H''_\alpha \\ \text{(iii)} & N^A_{2m} = N^B_{2m}. & \end{array}$$

Assume first that ($**$)(i) holds for at least an index h. In this case we can reproduce the same argument of 7.II and use Table II_1 (the only thing we want to point out is that in the actual case we do know that $e^A_{2m} > 0$).

Therefore we may assume that

$$M^A_{h'm} > M^B_{h'm} \qquad \text{for every } h' \in H_\beta$$

and ($**$) is:

$$\text{(ii)} \qquad N^A_{hm} = N^B_{hm} \qquad \text{for some } h \text{ in } H''_\alpha.$$

TABLE XI_1 for $(\mathscr{D}', \mathscr{D})$

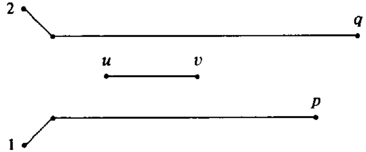
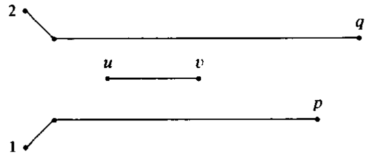
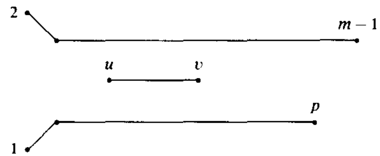
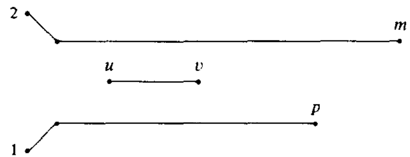
Degeneration	Factors	Type	Obstruction indices	
$\mathscr{D}'$ $B' \leqslant_{\mathscr{D}'} A' < A'$	 $p \leqslant q \leqslant m-1$	XI	$(a, b) \in \mathrm{ob}(\mathscr{D}')$	
			(α) $\begin{cases} a \in \{2\} = H'_\alpha, \\ a \in [0, u-1] = H''_\alpha, \end{cases}$	$b \in [p+1, q] = K'_\alpha$ $b \in [v+1, q] = K'_\alpha$
			(β) $a \in [v+1, q] = H_\beta,$	$b \in [p+1, m-1] = K_\beta$
$\mathscr{D}$ $B \nleqslant_{\mathscr{D}} A < A$		XI	$(h, \kappa) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$	
			(β) $h \in H_\beta,$	$\kappa = m$

TABLE XI_2 for $(\mathscr{D}', \mathscr{D})$

Degeneration	Factors	Type	Obstruction indices	
$\mathscr{D}'$ $B' \leqslant_{\mathscr{D}'} A' < A'$	 $p \leqslant m-1$	XI	$(a, b) \in \mathrm{ob}(\mathscr{D}')$	
		(α)	$\begin{cases} a \in \{2\} = H'_\alpha, \\ a \in [0, u-1] = H''_\alpha, \end{cases}$	$b \in [p+1, m-1] = K'_\alpha$ $b \in [v+1, m-1] = K''_\alpha$
		(β)	$a \in [v+1, m-1] = H_\beta,$	$b \in [p+1, m-1] = K_\beta$
$\mathscr{D}$ $B \nleqslant_{\mathscr{D}} A < A$		XI	$(h, \kappa) \in \mathrm{ob}(\mathscr{D}) - \mathrm{ob}(\mathscr{D}')$	
		(α)	$h \in H'_\alpha \cup H''_\alpha,$	$\kappa = m$
		(β)	$h \in H_\beta,$	$\kappa = m$

We choose such an index h and we define the following sets:

$$\Lambda_0 = \{E_{xy} : e^A_{xy} > 0, 3 \leqslant x \leqslant u-1, v+1 \leqslant y \leqslant m-1\}$$

$$\Lambda_1 = \{E_{2y} : e^A_{2y} > 0, p+1 \leqslant y \leqslant m-1\}$$

$$\Lambda_2 = \{F_{xy} : f^A_{xy} > 0, v+1 \leqslant x \leqslant m-1, p+1 \leqslant y \leqslant m\}$$

$$\Lambda_3 = \{E_{2z} : e^A_{2z} > 0, v+1 \leqslant z \leqslant p\}$$

$$\Lambda_4 = \{F_{xy} : f^A_{xy} > 0, v+1 \leqslant x < y \leqslant p\}.$$

Note that $\Lambda_0 = \varnothing$, otherwise we could perform a degeneration $\mathscr{D}^*$ of type I on the factors $E_{xy} \oplus E_{uv}$, $E_{xy} \in \Lambda_0$, against $(*)$. Similarly $\Lambda_1 = \varnothing$, otherwise we could perform $\mathscr{D}^*$ on $E_{2y} \oplus E_{uv} \oplus E_{1p}$, of type XI.

We set $\Lambda = \bigcup_{\tau=2}^{4} \Lambda_\tau \cup \{E_{1p}\}$ and at least a factor of A belongs to Λ (in fact by assumption $e^A_{1p} > 0$). We deduce Table XI_2 for $\tilde{\mathscr{D}}$.

Proof of ($\square$). If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $s = m$, ($\square$) holds as $r \in H_\beta$.

For the remaining pairs (r, s) of type (β) we consider the identity (3) of Lemma 6.2 and the permutation $\sigma = (1, 2)$. ($\circ\circ\circ$) holds, in fact:

$(e_{2s} + \cdots + e_{2\,m-1})^A = 0$ in 2 as z is max $(z \leqslant y < s)$ and $\Lambda_1 = \varnothing$; in 2′ as z is max and $\Lambda_1 = \varnothing$ in 3 as $\Lambda_3 \cap \Lambda_1 = \varnothing$;

$((e_{3r} + \cdots + e_{3\,m-1}) + \cdots + (e_{hr} + \cdots + e_{h\,m-1}))^A = 0$ as $\Lambda_0 = \varnothing$;

$((f_{rs} + \cdots + f_{rm}) + \cdots + (f_{s\,s+1} + \cdots + f_{sm}) + \cdots + f_{m-1\,m})^A = 0$ in 2 as y is max and $\Lambda_2 = \varnothing$; in 2′ as y is max $(y < z < s)$ and $\Lambda_2 = \varnothing$; in 3 as $\Lambda_4 \cup \Lambda_2 = \varnothing$.

(5.5)(β) follows now from ($\circ\circ\circ$), $(**)$(ii), and the inequalities:

$$N^A_{2s} \geqslant N^B_{2s}, \quad N^A_{1n} \geqslant N^B_{1n} \quad (B < A); \quad N^A_{hr} > N^B_{hr} \quad ((h, r) \in \mathrm{ob}(\mathscr{D}')).$$

If (r, s) is of type (α) we use the identity 2 of Lemma 6.2 and the permutation $\sigma = (1, 2)$. ($\circ\circ\circ$) holds, in fact:

$(e_{2s} + \cdots + e_{2\,m-1})^A = 0$, in 2′ as z is max and $\Lambda_1 = \varnothing$; in 3 as $\Lambda_3 \cup \Lambda_1 = \varnothing$;

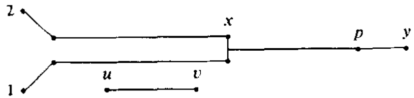
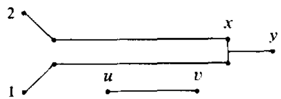
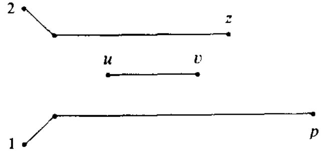
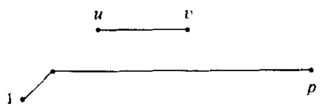
$((e_{3s} + \cdots + e_{3\,m-1}) + \cdots + (e_{hs} + \cdots + e_{h\,m-1}))^A = 0$ as $\Lambda_0 = \varnothing$;

$(f_{s\,s+1} + \cdots + f_{m-1\,m})^A = 0$ in 2′ as y in max $(y < z < s)$, and $\Lambda_2 = \varnothing$; in 3 as $\Lambda_4 = \varnothing$.

(5.5)(α) follows from ($\circ\circ\circ$), $(**)$(ii), and the inequalities:

$$N^A_{1m} \geqslant N^B_{1m} \quad (B < A); \quad N^A_{hs} > N^B_{hs} \quad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

TABLE XI_2 for $\tilde{\mathscr{D}}$: Assumption (**)(ii)

Column 1					Column 2	
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{S}}) - \mathrm{ob}(\mathscr{S}')$	
$\Lambda_2 \neq \varnothing$	$F_{xy} \in \Lambda_2$	1		V	(β) $r \in [v+1, x]$,	$s = m$
$\Lambda_2 = \varnothing$ $\Lambda_3 \cup \Lambda_4 \neq \varnothing$	$E_{2z} \in \Lambda_3$ z max $F_{xy} \in \Lambda_4$ y max	2	 $z \leqslant y$ (or $\Lambda_3 = \varnothing$)	V	(β) $\begin{cases} r \in [v+1, x], \\ r \in [v+1, x], \end{cases}$	$s \in [y+1, p]$ $s = m$
		2′	 $z > y$ (or $\Lambda_4 = \varnothing$)	XI	(α) $r = 1$, (β) $\begin{cases} r \in [v+1, p-1], \\ r \in [v+1, p], \end{cases}$	$s \in [z+1, p]$ $s \in [z+1, p]$ $s = m$
$\bigcup_{\tau=2}^{4} \Lambda_\tau = \varnothing$	$E_{1p} \in \Lambda$	3		II	(α) $r = 1$, (β) $\begin{cases} r \in [v+1, p-1], \\ r \in [v+1, p], \end{cases}$	$s \in [v+1, p]$ $s \in [v+2, p]$ $s = m$

Proposition (5.3) is now proved under the assumption $(**)$(i) and (ii), therefore we may assume:

$$M^A_{h'm} > M^B_{h'm} \qquad \text{for every } h' \in H_\beta; \qquad N^A_{h'm} > N^B_{h'm} \qquad \text{for every } h' \in H''_\beta$$

and the assumption $(**)$ is:

$$\text{(iii)} \qquad N^A_{2m} = N^B_{2m}.$$

We have (cf. assumption $(***)$)

$$(\circ) \qquad N^A_{1m} - N^A_{2m} + e^A_{2m} > N^B_{1m} - N^B_{2m} + e^B_{2m} \geqslant 0$$

as we know that $e^A_{2m} > 0$, and from the identity (6) of Lemma 6.1 we have

$$(\circ\circ) \qquad e^A_{1m} > 0.$$

The degeneration $\tilde{\mathscr{D}}$ we consider is shown in Table XI_2 assumption $(**)$(iii). ($\square$) holds as $r \in H''_\alpha$.

To end the discussion we only have to assume for the pairs $(\mathscr{D}', \mathscr{D})$ the situation shown in Table XI_3.

The assumption $(**)$ now is:

$$(**) \qquad N^A_{hm} = N^B_{hm} \qquad \text{for some } h \in H_\alpha.$$

We choose such an index h. We have

$$(\circ) \qquad N^A_{h\ m-1} + N^A_{hm} > N^B_{h\ m-1} - N^B_{hm} \geqslant 0;$$

$$(\circ\circ) \quad (e_{1\ m-1} + e_{2\ m-1} + e_{3\ m-1} + \cdots + e_{h\ m-1} + f_{0\ m-1} + \cdots + f_{m-2\ m-1} + f_{m-1\ m})^A > 0$$

(cf. identity (1) of Lemma 6.1).

We consider the following sets:

$$A_0 = \{E_{xy} : e^A_{xy} > 0, 3 \leqslant x \leqslant u-1, v+1 \leqslant y \leqslant m-1\}$$

$$A_1 = \{E_{1w} : e^A_{1w} > 0, v+1 \leqslant w \leqslant m-1\}$$

$$A_2 = \{E_{2z} : e^A_{2z} > 0, v+1 \leqslant z \leqslant m-1\}$$

$$A_3 = \{F_{xy} : f^A_{xy} > 0, v+1 \leqslant x < y \leqslant m\}$$

$$A_4 = \{F_{vy} : f^A_{vy} > 0, v+1 \leqslant y \leqslant m-1\}$$

$$A_5 = \{F_{xy} : f^A_{xy} > 0, 0 \leqslant x \leqslant v-1, v+1 \leqslant y \leqslant m-1\}.$$

We have $A_0 = \varnothing$, not to go against $(*)$. We set $A = \bigcup_{\tau=1}^{5} A_\tau$, and we have $A \neq \varnothing$, as a consequence of $(\circ\circ)$.

TABLE $\mathrm{XI}_{2'}$ for $\tilde{\mathscr{D}}$: Assumption (**)(iii)

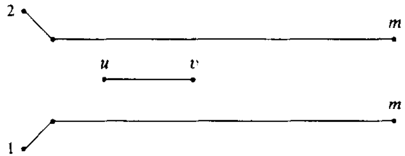
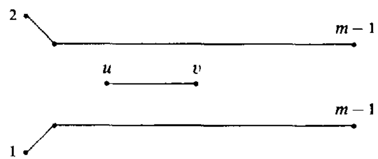
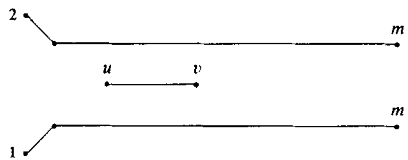
$\tilde{\mathscr{D}} : B \leqslant \tilde{\mathscr{D}} A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$
	XI	$r \in [0, u-1]$, $s = m$

TABLE XI_3 for $(\mathcal{D}', \mathcal{D})$

Degeneration	Factors	Type	Obstruction indices
$\mathcal{D}'$ $B' \leqslant \mathcal{D}'A' < A'$		XI	$(a, b) \in \mathrm{ob}(\mathcal{D}')$ (α) $a \in [0, u-1] = H_\alpha$, $b \in [v+1, m-1] = K_\alpha$
$\mathcal{D}$ $B \nleqslant \mathcal{D}A < A$		XI	$(h, \kappa) \in \mathrm{ob}(\mathcal{D}) - \mathrm{ob}(\mathcal{D}')$ (α) $h \in H_\alpha$, $\kappa = m$

Explanation of Table XI_3 *for* $\tilde{\mathscr{D}}$. If $A_1 \neq \varnothing$, $A_2 \neq \varnothing$, $A_3 \neq \varnothing$, we choose factors $E_{1w} \in A_1$, $E_{2z} \in A_2$, $F_{xy} \in A_3$ with the property that w, z, x are maximum. We may also assume that if $A_1 \neq \varnothing$ and $A_2 \neq \varnothing$ then $w \leqslant z$, and if $A_2 = \varnothing$ then $A_1 = \varnothing$.

If $A_1 \neq \varnothing$, $A_2 \neq \varnothing$, $A_3 \neq \varnothing$ and $y \leqslant w$ (or if $A_3 = \varnothing$ and $A_1 \neq \varnothing$, $A_2 \neq \varnothing$) we perform the degeneration 1; if $y > w$ (or $A_1 = \varnothing$, $A_3 \neq \varnothing$) we perform the degeneration $1'$; if $A_2 \neq \varnothing$ and $A_1 \cup A_3 = \varnothing$ we perform $1''$. The case we perform 2 or 3 is clearly indicated in the table.

Proof of ($\square$). If $(2, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ we consider the identity (2) of Lemma 6.2 ($\circ\circ\circ$) holds, in fact:

$(e_{1s} + \cdots + e_{1\,m-1})^A = 0$, in 1 as w is max; in $1''$, 2, 3, as $A_1 = \varnothing$;

$((e_{3s} + \cdots + e_{3\,m-1}) + \cdots + (e_{hs} + \cdots + e_{h\,m-1}))^A = 0$ as $A_0 = \varnothing$;

$((f_{s\,s+1} + \cdots + f_{sm}) + \cdots + f_{m-1\,m})^A = 0$ in 1 as y is max ($y \leqslant w < s$); in $1''$, 2, 3 as $A_3 = \varnothing$.

(5.3)(α) follows from ($**$), ($\circ\circ\circ$), and the inequalities:

$$N^A_{2m} \geqslant N^B_{2m} \qquad (B < A); \qquad N^A_{hs} > N^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

If $(1, r) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ we use the identity (2) of Lemma 6.2 transformed via the permutation $\sigma = (1, 2)$, and we get the same conclusion. If $(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$ is of type (β) and $r \in [x+1, v]$, $s \in [v+1, y]$ (degeneration 3), we consider the identity (5) of Lemma 6.2. ($\circ\circ\circ$) holds, in fact:

$((e_{1s} + \cdots + e_{1\,m-1}) + (e_{2s} + \cdots + e_{2\,m-1}) + (e_{3s} + \cdots + e_{3\,m-1}) + \cdots$
$+ (e_{hs} + \cdots + e_{h\,m-1}))^A = 0$ as $A_1 = A_2 = A_0 = \varnothing$;

$(f_{rs} + \cdots + f_{r\,m-1}) + \cdots + (f_{s-1\,s} + \cdots + f_{s-1\,m-1}) + (2f_{s\,s+1} + \cdots$
$+ 2f_{s\,m-1} + f_{sm}) + \cdots + f_{m-1\,m}))^A = 0$ as x in max; $A_4 = A_3 = \varnothing$.

(5.3)(β) follows from ($**$), ($\circ\circ\circ$), and the inequalities:

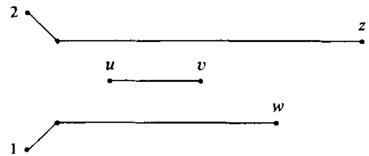
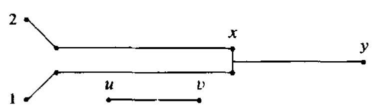
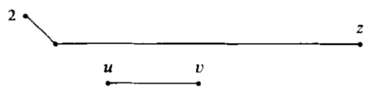
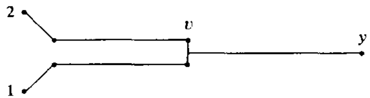
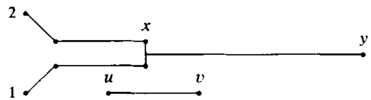
$$M^A_{rm} \geqslant M^B_{rm} \qquad (B < A); \qquad N^A_{hs} > N^B_{hs} \qquad ((h, s) \in \mathrm{ob}(\mathscr{D}')).$$

For the remaining pairs (r, s) of type (β) we use the identity (3) of Lemma 6.2. ($\circ\circ\circ$) holds, in fact:

$(e_{1s} + \cdots + e_{1\,m-1})^A = 0$, in 1 as w is max; in $1''$ as w is max ($w < y < s$); in $1''$, 2, 3 as $A_1 = \varnothing$;

$((e_{3r} + \cdots + e_{3\,m-1}) + \cdots + (e_{hr} + \cdots + e_{h\,m-1}))^A = 0$, as $A_0 = \varnothing$;

TABLE XI_3 for $\tilde{\mathscr{D}}$

Column 1				Column 2		
Choice in Λ			$\tilde{\mathscr{D}}: B \leqslant \tilde{\mathscr{D}}A < A$	Type	$(r, s) \in \mathrm{ob}(\tilde{\mathscr{D}}) - \mathrm{ob}(\mathscr{D}')$	
$\bigcup_{\tau=1}^{3} \Lambda_\tau \neq \varnothing$	$E_{1w} \in \Lambda_1$ w max $E_{2z} \in \Lambda_2$ z max	1		XI	(α) $r = 2$, (β) $r \in [v+1, z-1]$,	$s \in [w+1, z]$ $s \in [w+1, z]$
			$y \leqslant w$ (or $\Lambda_3 = \varnothing$, $\Lambda_1 \neq \varnothing$, $\Lambda_2 \neq \varnothing$)			
	$F_{xy} \in \Lambda_3$ y max $w \leqslant z$	1′		V	(β) $r \in [v+1, x]$,	$s \in [y+1, m]$
			$w < y$ (or $\Lambda_1 = \varnothing$, $\Lambda_3 = \varnothing$)			
		1″		II	(α) $r = 2$, (β) $r \in [v+1, z]$,	$s \in [v+1, z]$ $s \in [v+2, m]$
			$\Lambda_1 \cup \Lambda_3 = \varnothing$			
$\bigcup_{\tau=1}^{3} \Lambda_\tau = \varnothing$ $\Lambda_4 \neq \varnothing$	$F_{vy} \in \Lambda_4$	2		IX	(α) $r \in \{1, 2\}$, (β) $r \in [v+1, y-1]$,	$s \in [v+1, y]$ $s \in [v+2, y]$
$\bigcup_{\tau=1}^{4} \Lambda_\tau = \varnothing$ $\Lambda_5 \neq \varnothing$	$F_{xy} \in \Lambda_5$ x max	3		III	(α) $r \in \{1, 2\}$, (β) $\begin{cases} r \in [x+1, v], \\ r \in [v+1, y], \end{cases}$ (γ) $r \in [v+1, y-1]$,	$s \in [v+1, y]$ $s \in [v+1, y]$ $s \in [y+1, m]$ $s \in [v+2, y]$

$((f_{rs}+\cdots+f_{rm})+\cdots+(f_{s\,s+1}+\cdots+f_{m-1\,m}))^A=0$, in 1 as y is max; $(y\leqslant w<s)$; in 1′ as y is max; in 1″, 2, 3 as $\Lambda_3=\varnothing$.

(5.3)(β) follows (∗∗), (∘∘∘), and the inequalities:

$$N^A_{1s}\geqslant N^B_{1s},\quad N^A_{2s}\geqslant N^B_{2s}\quad (B<A);\quad N^A_{hs}>N^B_{hr}\quad ((h,r)\in \mathrm{ob}(\mathscr{D}')).$$

If $(r,s)\in \mathrm{ob}(\tilde{\mathscr{D}})-\mathrm{ob}(\mathscr{D}')$ is of type (γ) (degeneration 3), we consider the identity (3) of Lemma (6.2), and (∘∘∘) holds as $\Lambda_0=\Lambda_1=\Lambda_3=\varnothing$. (5.3)($\gamma$) is now a consequence of (∗∗), (∘∘∘), and the inequalities:

$$N^A_{2m}\geqslant N^B_{2m}\quad (B<A);\quad N^A_{hr}>N^B_{hr}\quad ((h,r)\in \mathrm{ob}(\mathscr{D}'));\quad N^A_{1s}>N^B_{1s}$$

(proved above).

8. Proof of Proposition 5.3, Step 2

Recall that our assumptions now are (cf. Section 5, step 2):

$$\begin{cases} N^A_{rs}=N^B_{rs} & \text{for every } r\in [0,m-1]\cup\{1,2\},\ r\leqslant s<m\\ M^A_{rs}=M^B_{rs} & \text{for every } r\in [3,m-1],\ r<s<m, \end{cases} \tag{8.1}$$

i.e., $B'=A'$, and $B<A$ implies that, for at least an index, one of the following strict inequalities holds:

$$\begin{aligned} &\text{(i)}\quad M^A_{m-1\,m}>M^B_{m-1\,m}\\ &\text{(ii)}\quad M^A_{h\,m}>M^B_{h\,m},\qquad h\in[3,m-2]\\ &\text{(iii)}\quad N^A_{\tau\,m}>N^B_{\tau\,m},\qquad \tau=1,2\\ &\text{(iv)}\quad N^A_{0\,m}>N^B_{0\,m}\\ &\text{(v)}\quad N^A_{h\,m}>N^B_{h\,m},\qquad h\in[3,m-1]. \end{aligned} \tag{8.2}$$

If (∗∗)(i) holds, we consider the following identity:

$$M_{m-1\,m}-N_{0\,m-1}=f_{0\,m}+f_{3\,m}+\cdots+f_{m-2\,m}.$$

We have $[M_{m-1\,m}-N_{0\,m-1}]^A>[M_{m-1\,m}-N_{0\,m-1}]^B\geqslant 0$ and therefore:

$$[f_{0\,m}+f_{3\,m}+\cdots+f_{m-2\,m}]^A>0.$$

The set $\Lambda_0=\{F_{xm}: f^A_{xm}>0,\ 0\leqslant x\leqslant m-2\}$ is not empty as a consequence of (∘∘).

We consider also the following sets:

$$A_1 = \{E_{1\ m-1} : e^A_{1\ m-1} > 0\}$$
$$A_2 = \{E_{2\ m-1} : e^A_{2\ m-1} > 0\}$$
$$A_3 = \{F_{m-1\ m} : f^A_{m-1\ m} > 0\}$$
$$A_4 = \{E_{z\ m-1} : e^A_{z\ m-1} > 0, 3 \leqslant z \leqslant m-1\}.$$

We set $A = \bigcup_{\tau=0}^4 A_\tau \neq \varnothing$ and we define the degenerations $\mathscr{D}$ (see Table 8_1 for $\mathscr{D}$). Note that the pairs $(r, s) \in \mathrm{ob}(\mathscr{D})$ are such that $s = m$. We have to prove now that for $(r, m) \in \mathrm{ob}(\mathscr{D})$ of type (α), $N^A_{rm} > N^B_{rm}$, and for $(r, m) \in \mathrm{ob}(\mathscr{D})$ of type (β), $M^A_{rm} > M^B_{rm}$. The last inequality holds for $r = m-1$ as we are assuming (8.2)(i).

If $(r, m) \in \mathrm{ob}(\mathscr{D})$ is the type (β) and $r \leqslant m-2$, we consider the identity

$$M_{rm} = M_{m-1\ m} + N_{0\ r} - N_{0\ m-1} - [f_{r\ m} + \cdots + f_{m-2\ m}].$$

We have

$$(\circ\circ\circ) \qquad [f_{r\ m} + \cdots + f_{m-2\ m}]^A = 0 \qquad \text{as } x \text{ is maximum.}$$

We use the same identity for the factors A and B and we deduce $M^A_{rm} > M^B_{rm}$, as a consequence of (8.2)(i), $(\circ\circ\circ)$, and the equalities:

$$N^A_{0r} = N^B_{0r}, \qquad N^A_{0\ m-1} = N^B_{0\ m-1} \qquad \text{(cf. (8.1))}.$$

If $(2, m) \in \mathrm{ob}(\mathscr{D})$, we consider the identity

$$N_{2m} = M_{m-1\ m} - N_{1\ m-1} - e_{2\ m-1}.$$

We have:

$$(\circ\circ\circ) \qquad e^A_{2\ m-1} = 0 \qquad \text{as } A_2 = \varnothing.$$

Using the same identity for A and B we get $N^A_{2m} > N^B_{2m}$, as a consequence of (8.2)(i) and the equality $N^A_{1\ m-1} = N^B_{1\ m-1}$ (cf. 8.1).

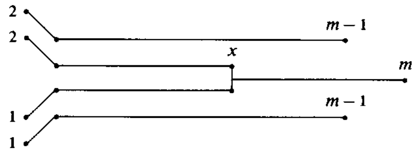
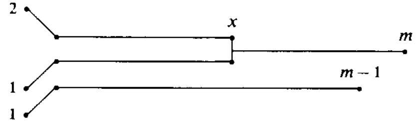
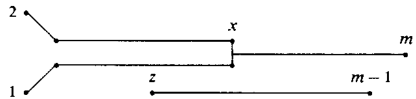
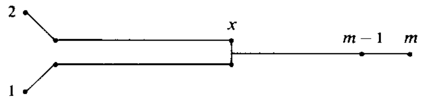
The same argument holds for $(1, m) \in \mathrm{ob}(\mathscr{D})$, using the permutation $\sigma = (1, 2)$. If $(r, m) \in \mathrm{ob}(\mathscr{D})$ is of type (α) and $r \neq 1, 2$ (degenerations 4, 5), we consider the identity

$$N_{rm} = N_{r\ m-1} + M_{m-1\ m} - N_{1\ m-1} - N_{2\ m-1} - [e_{1\ m-1} + e_{2\ m-1} + e_{3\ m-1} + \cdots + e_{r\ m-1} + f_{m-1\ m}].$$

We have

$$(\circ\circ\circ) \qquad [\cdots]^A = 0,$$

TABLE 8_1 for $\mathscr{D}$: Assumption (8.2)(i)

Choice in Λ			$\mathscr{D}: B \leqslant \mathscr{D}A < A$	Type	$(r,s) \in \mathrm{ob}(\mathscr{D})$	
$\Lambda_1 \neq \varnothing$ $\Lambda_2 \neq \varnothing$	$E_{1m-1} \in \Lambda_1$ $E_{2m-1} \in \Lambda_2$ $F_{xm} \in \Lambda_0$ x max	1		X	(β) $r \in [x+1, m-1]$,	$s = m$
$\Lambda_1 \neq \varnothing$ $\Lambda_2 = \varnothing$	$E_{1m-1} \in \Lambda_1$ $F_{xm} \in \Lambda_0$ x max	2		IV	(α) $r = 2$, (β) $r \in [x+1, m-1]$,	$s = m$ $s = m$
$\Lambda_1 \cup \Lambda_2 = \varnothing$ $\Lambda_3 \neq \varnothing$	$F_{m-1\,m} \in \Lambda_3$	2		IX	(α) $r = 2$,	$s = m$
$\bigcup_{\tau=1}^{3} \Lambda_\tau = \varnothing$ $\Lambda_4 \neq \varnothing$	$E_{z\,m-1} \in \Lambda_4$ z min $F_{xy} \in \Lambda_0$ x max	4		III	(α) $\begin{cases} r \in \{1, 2\}, \\ r \in [0, z-1], \end{cases}$ (β) $r \in [x+1, m-1]$,	$s = m$ $s = m$ $s = m$
$\bigcup_{\tau=1}^{4} \Lambda_\tau = \varnothing$	$F_{xy} \in \Lambda_0$ x max	5		III′ $t = m+1$	(α) $\begin{cases} r = \{1, 2\}, \\ r = [0, m-1], \end{cases}$ (β) $r \in [x+1, m-1]$,	$s = m$ $s = m$ $s = m$

in fact

$e^A_{1\ m-1} = e^A_{2\ m-1} = 0$ as $A_1 \cup A_2 = \varnothing$;

$(e_{3\ m-1} + \cdots + e_{r\ m-1})^A = 0$ in 4 as z is min, in 5 as $A_4 = \varnothing$;

$f^A_{m-1\ m} = 0$ as $A_3 = \varnothing$.

Proposition 5.3 is now proved under the assumption 8.2)(i); it follows that we may assume:

$$M^A_{m-1\ m} = M^B_{m-1\ m}$$

and (8.2)

(ii) $\qquad M^A_{h\ m} > M^B_{h\ m} \qquad$ for some $h \in [3, m-2]$.

Among all the indices such that (8.)(ii) holds we choose an index h maximum, and therefore $M^A_{h+1\ m} = M^B_{h+1\ m}$. We consider the following identities:

$$M_{hm} - N_{0h} = f_{0m} + f_{3m} + \cdots + f_{h-1\ m}$$

$$M_{h+1\ m-1} - M_{h+1\ m} - M_{h\ m-1} + M_{h\ m} = f_{h\ m-1}.$$

For the left-hand side of such identities we have:

$$(\text{L.H.S.})^A > (\text{L.H.S.})^B \geqslant 0$$

(we use here the maximality of the index h). Therefore we have:

$$(f_{0m} + f_{3m} + \cdots + f_{h-1\ m})^A > 0; \qquad f^A_{h\ m-1} > 0.$$

It follows that $F_{h,m-1}$ is a factor in A and the set

$A = \{F_{xm} : f^A_{xm} > 0, 0 \leqslant x \leqslant h-1\}$ is non-empty. $\mathscr{D}$ is defined in Table 8_2.

We already know that $M^A_{h\ m} > M^B_{hm}$ (assumption (8.2)(ii), and if $(r, m) \in \text{ob}(\mathscr{D})$, $x + 1 \leqslant r \leqslant h-1$, we consider the identity

$$M_{rm} = M_{hm} + N_{0r} - N_{0h} - [f_{rm} + \cdots + f_{h-1\ m}].$$

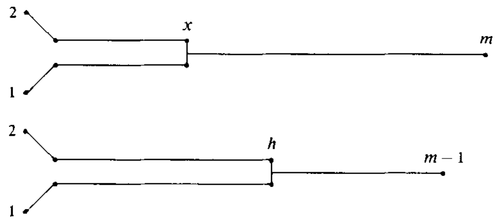
We have $[\cdots]^A = 0$ as x is maximum. The inequality $M^A_{rm} > M^B_{rm}$ now follows from (8.2)(ii) (see Table XII_2 for $\mathscr{D}$) and the equalities

$$N^A_{0h} = N^B_{0h}, \qquad N^A_{0r} = N^B_{0r} \qquad (\text{cf. } 8.1).$$

Proposition 5.3 is now proved under the assumptions (8.2)(i) and (ii), therefore we may assume:

$$M^A_{h'\ m-1} = M^B_{h'\ m-1} \qquad \text{for every } h' \in [3, m-1]$$

TABLE 8_2 for $\mathscr{L}$: Assumption (8.2)(ii)

Choice in Λ	$\mathscr{L}: B \leqslant \mathscr{L}A < A$	Type	$(r, s) \in \mathrm{ob}(\mathscr{L})$
$F_{xm} \in \Lambda$ x max		VI	(β) $r \in \lvert x+1, h \rvert$, $s = m$

and (8.2)

(iii) $$N^A_{1m} > N^B_{1m} \qquad \text{(we choose here } \tau = 1\text{)}.$$

We consider the identity

$$N_{1m} + N_{0\ m-1} - M_{m-1\ m} = e_{1\ m} + f_{m-1\ m}.$$

For its left-hand side we have $[\text{L.H.S.}]^A > [\text{L.H.S.}]^B \geqslant 0$, therefore we have

(∘∘∘) $$e^A_{1m} + f^A_{m-1\ m} > 0$$

We consider

$\Lambda_3 = \{F_{m-1\ m} : f^A_{m-1\ m} > 0\}$

$\Lambda_5 = \{E_{1\ m} : e^A_{1\ m} > 0\}$

$\Lambda_4 = \{E_{z\ m-1} : e^A_{z\ m-1} > 0, 3 \leqslant z \leqslant m-1\}$.

Note that from (∘∘∘) it follows $\Lambda_3 \cup \Lambda_5 \neq \varnothing$. We set

$$\Lambda = \bigcup_{\tau=3}^{5} \Lambda_\tau \neq \varnothing.$$

We already know that $N^A_{1m} > N^B_{1m}$ (assumption (8.2)(iii); see Table 8_3 for $\mathscr{D}$). For the remaining pairs $(r, m) \in \text{ob}(\mathscr{D})$ (degenerations 2, 3) we consider the identity:

$$N_{rm} = N_{r\ m-1} - N_{0\ m-1} + N_{2\ m} + e_{1\ m} - [e_{3\ m-1} + \cdots + e_{r\ m-1}]$$

where $[\cdots]$ disappear if $r = 0$. We have:

(∘∘∘) $\quad [\cdots]^A = 0 \qquad$ in 2 as z is minimum; in 3 as $\Lambda_4 = \varnothing$.

We use the same identity for A and B, and we get $N^A_{rm} > N^B_{rm}$, as a consequence of (8.1), (∗∗∗) (∘∘∘), and the inequalities:

$$N^A_{2m} \geqslant N^B_{2m} \qquad (B < A); \qquad e^A_{1,m} > 0 \qquad (\Lambda_2 \neq \Phi).$$

Proposition 5.3 is now proved under the assumptions (8.2)(i), (ii), (iii); therefore we may assume:

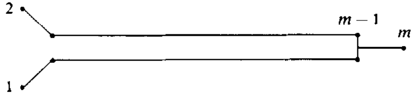
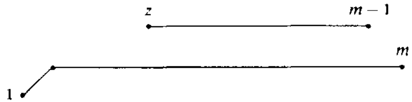
$$M^A_{h'm} = M^B_{h'm} \qquad \text{for every } h' \in [3, m-1]$$

$$N^A_{\tau\ m} = N^B_{\tau\ m}, \qquad \tau = 1, 2$$

and (8.2)

(iv) $$N^A_{0m} > N^B_{0m}.$$

TABLE 8_3 for $\mathscr{D}$: Assumption (8.2)(iii)

Choice in Λ			$\mathscr{D}: B \leqslant \mathscr{D}A < A$	Type	$(r, s) \in \mathrm{ob}(\mathscr{D})$	
$\Lambda_3 \neq \varnothing$	$F_{m-1m} \in \Lambda_3$	1		IX	(α) $r = 1$,	$s = m$
$\Lambda_3 = \varnothing$ $\Lambda_4 \neq \varnothing$ $\Lambda_5 \neq \varnothing$	$E_{z\,m-1} \in \Lambda_4$ z min $E_{1m} \in \Lambda_5$	2		II	(α) $\begin{cases} r = 1, \\ r \in [0, z-1], \end{cases}$	$s = m$ $s = m$
$\Lambda_3 = \varnothing$ $\Lambda_4 = \varnothing$ $\Lambda_5 \neq \varnothing$	$E_{1m} \in \Lambda_5$	3		II′ $t = m - 1$	(α) $\begin{cases} r = 1, \\ r = [0, m-1], \end{cases}$	$s = m$ $s = m$

We consider the identities:

$$N_{0m} - N_{1m} = e_{2m}; \qquad N_{0m} - N_{2m} = e_{1m}.$$

From (8.2)(iv) it follows that $e^A_{2m} > 0$, $e^A_{1m} > 0$, i.e., in A there are the factors E_{2m} and E_{1m}. We consider the set

$\Lambda = \{E_{x\ m-1} : e^A_{x\ m-1} > 0, 3 \leqslant x \leqslant m-1\}$.

We already know that $N^A_{0m} > N^B_{0m}$ (assumption (8.2); see Table 8_4 for $\mathscr{D}$); to prove that $N^A_{rm} < N^B_{rm}$ for the remaining pairs we consider the identity:

$$N_{rm} = N_{r\ m-1} - N_{0\ m-1} + N_{0m} + [e_{3\ m-1} + \cdots + e_{r\ m-1}].$$

(ooo) $\qquad [\cdots]^A = 0 \qquad$ as x is minimum (or $\Lambda = \varnothing$).

The claim follows from (8.2)(iv), and from the equalities:

$$N^A_{r\ m-1} = N^B_{r\ m-1}, \qquad N^A_{0\ m-1} = N^B_{0\ m-1}.$$

Proposition 5.3 is now proved under the assumption (8.2)(i),..., (iv); therefore we may assume:

$$M^A_{h'\ m} = M^B_{h'\ m} \qquad \text{for all } h' \in [3, m-1]$$

$$N^A_{h'\ m} = N^B_{h'\ m} \qquad \text{for all } h' = 0, 1, 2$$

and (8.2) is

(v) $\qquad N^A_{h\ m} > N^B_{h\ m} \qquad$ for some $h \neq 0, 1, 2$.

Among the indices such that (8.2)(v) holds we choose an index h which is minimum possible; it follows that we also have $N^A_{h-1\ m} = N^B_{h-1\ m}$.

From the identity $N_{hm} - N_{h-1\ m} = e_{hm}$ we deduce that $e^A_{hm} > 0$, i.e., E_{hm} is a factor of A. We consider the set

$\Lambda = \{E_{x\ m-1} : e^A_{x\ m-1} > 0, h+1 \leqslant x \leqslant m-1\}$.

We already know that $N^A_{hm} > N^B_{hm}$ (assumption (8.2)(v); see Table 8_5 for $\mathscr{D}$). For the remaining pairs (r, m) we consider the identity:

$$N_{rm} = N_{r\ m-1} - N_{h\ m-1} + N_{hm} - [e_{h+1\ m-1} + \cdots + e_{r\ m-1}].$$

We have

(ooo) $\qquad [\cdots]^A = 0 \qquad$ as x is minimum (or $\Lambda = \varnothing$).

TABLE 8_4 for $\mathscr{D}$: Assumption (8.2)(iv)

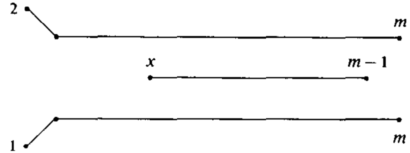
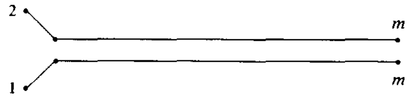
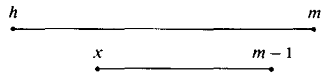
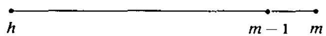
			$\mathscr{D}: B \leqslant \mathscr{D}A < A$	Type	$(r, s) \in \mathrm{ob}(\mathscr{D})$
$\Lambda \neq \varnothing$	$E_{1m-1} \in \Lambda$ x min	1		XI	(α) $r \in [0, x-1]$, $s = m$
$\Lambda = \varnothing$		2		XI′ $t = m - 1$	(α) $r \in [0, m-1]$, $s = m$

TABLE 8_5 for $\mathscr{L}$: Assumption (8.2)(v)

			$\mathscr{L}: B \leqslant \mathscr{L}A < A$	Type	$(r, s) \in \mathrm{ob}(\mathscr{L})$
$\Lambda \neq \varnothing$	$E_{x\,m-1} \in \Lambda$ x min	1		I	(α) $r \in [h, x-1]$, $s = m$
$\Lambda = \varnothing$		2		I′ $t = m - 1$	(α) $r \in [h, m-1]$, $s = m$

It follows that

$$N^A_{rm} > N^B_{rm}$$

as a consequence of (∘∘∘), (8.3)(v), and the equalities:

$$N^A_{r\,m-1} = N^B_{r\,m-1}, \qquad N^A_{h\,m-1} = N^B_{h\,m-1}.$$

Proposition 5.3 is now completely proved.

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